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Semilocal Convergence Newton Method Applied to Kepler Equation: New Results

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Authors' contributions

This work was carried out in collaboration between all authors. Authors MAD and JMG designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author EV managed the analyses of the study and literature searches. All authors read and approved the final manuscript.

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Abstract

The aim of this paper is to consider as a starting point the value $E_0 = \pi$ in the theorems of semilocal convergence of Kantorovich, Gutiérrez, $\alpha-$theory of Smale and the $\alpha-$theory of Wang-Zhao, to compare the convergence conditions obtained. Once set E_0, one should calculate the parameters listed in the statement of these theorem. So, we will generalize the study of Diloné-Gutiérrez for the case $E_0 = M$.
Numeric and graphic calculations were obtained by applying Mathematica V10.

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1 Introduction

We must keep in mind that the convergence of an iterative method is not always going to happen. For this to happen, a series of conditions have to be given on the function, the starting point or on the root. In this sense, 3 types of convergence results have been distinguished:

1. Local: conditions are given on the root.
2. Semilocal: conditions are given on the starting point.
3. Global: conditions are given over an interval.

Recall that the results of semilocal convergence, in addition to ensuring the convergence of Newton's method to a solution of the equation considered, provide results of existence and uniqueness of the solution.

In this article we analyze the behavior of the famous Kepler's equation,

$$f(E) = E - e \sin E - M, \tag{1.1}$$

where eccentricity $e \in (0, 1)$ and the mean anomaly $M \in [0, \pi]$ are known parameters. We have selected this equation as a function test in the analysis of various semilocal convergence results for the Newton-Raphson method, which when is applied to (1.1) gives the sequence

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0. \tag{1.2}$$

The main idea of the paper is to construct the corresponding majoring function for the mentioned theories for the case $E_0 = \pi$ and then to give conditions on the eccentricity e that guarantee that these majoring functions have real roots. In this way, we establish conditions for the solution for Kepler 's equation in terms of eccentricity e.

The authors consider the results given by Kantorovich [1], Gutiérrez [2], α-theory of Smale [3] and α-theory of Wang-Zhao [4]. In addition, the authors obtain new results of existence and uniqueness of solution for the equation (1.1). In this they generalized the study in [5] for the case $E_0 = M$.

We present now the semilocal convergence theorems that we use in this paper. Although they can be stated in a Banach space setting, we show here the version for real valued functions.The details of the demonstration of the previous theorems can be found in Dilоné [6].

Theorem 1.1 (**Kantorovich's Theorem**). *Let us consider $f : I \to \mathbb{R}$, where I is an open interval in $\mathbb{R}$, a differentiable function in I. Let us assume, without loss of generality, that $f(x_0) \neq 0$ and*

i) $x_0 \in I$.

ii) $f'(x_0) \neq 0$.

iii) $\left|\frac{f(x_0)}{f'(x_0)}\right| \leq \beta$.

iv) $\left|\frac{f'(x)-f'(y)}{f'(x_0)}\right| \leq \gamma|x - y|, \quad \forall x, y \in I$.

v) $h = \gamma\beta \leq 1/2$.

vi) $t^* = \frac{1-\sqrt{1-2h}}{2\beta}$ *it is such that* $(x_0 - t^*, x_0 + t^*) \subset I$.

Then, the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \tag{1.3}$$

is well defined and $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^* \in I$*, where* $f(x^*) = 0$*. In addition, this solution is located in the range*

$$[x_0 - t^*, x_0 + t^*]. \tag{1.4}$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \tag{1.5}$$

As it was pointed out by Argyros et al [7] the Lipschitz condiction in Kantorovich's theorem can be weakened by combining a center-Lipschitz with the usual Lipschitz condition.

Theorem 1.2 (Argyros-Hilout's Theorem). *Let us consider* $f : I \to \mathbb{R}$*, where* I *is an open interval in* $\mathbb{R}$ *and* f *a differentiable function defined in* I*. Let us suppose that there exist* $x_0 \in I$*, and* $\eta > 0$ *such that*

i) $\left|\frac{f(x_0)}{f'(x_0)}\right| \leq \eta$,

ii) $\left|\frac{f'(x)-f'(x_0)}{f'(x_0)}\right| \leq L_0|x - x_0|$ *for all* $x \in I$,

iii) $\left|\frac{f'(x)-f'(y)}{f'(x_0)}\right| \leq L|x - y|$ *for all* $x, y \in I$,

iv) $h_A = L_A\eta \leq \frac{1}{2}$*, where* $L_A = \frac{1}{8}\left(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}\right)$

v) $\{x \in \mathbb{R}, |x - x_0| \leq t^*\} \leq I$ *where* $t^* = \lim_{n\to\infty} t_n$ *and* t_n *is the scalar sequence given by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_2 = \eta + \frac{L_0\eta^2}{2(1 - L_0\eta)}, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0t_{n+1})} \quad \forall n = 1, 2, \ldots$$

Then, the Newton-Raphson method defined in (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^* \in I$*, where* $f(x^*) = 0$*.*

Theorem 1.3 (Gutiérrez's theorem or Kantorovich's theorem under strong conditions). *Let us consider* $f : I \to \mathbb{R}$*, where* I *is an open interval in* $\mathbb{R}$*, a differentiable function in* I*. Let us assume, without loss of generality, that* $f(x_0) \neq 0$ *and that*

i) $\left|\frac{f(x_0)}{f'(x_0)}\right| \leq a$.

ii) $\left|\frac{f''(x_0)}{f'(x_0)}\right| \leq b$.

iii) $\left|\frac{f''(x)-f''(x_0)}{f'(x_0)}\right| \leq K|x - x_0|$.

If the majorizing polynomial

$$\phi_G(t) = a - t + \frac{b}{2}t^2 + \frac{K}{6}t^3, \tag{1.6}$$

has two positive real roots t^* *and* t^{**}*, then the Newton-Raphson method defined in (1.3) well defined and* $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^* \in I$*, where* $f(x^*) = 0$*. This solution is located in (1.4) and it is unique in (1.5).*

Theorem 1.4 (Smale α-theory). *Let* $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *a continuous and differentiable function in an interval* D *where the following conditions hold:*

i) $\left|\frac{f(x_0)}{f'(x_0)}\right| \leq \beta$. $f'(x_0) \neq 0$.

ii) $\frac{1}{k!}\left|\frac{f^{(k)}(x_0)}{f'(x_0)}\right| \leq \gamma^{k-1}$, *for* $k \geq 2$.

iii) $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$.

If the majorizing function

$$\phi_S(t) = \beta - t + \sum_{k\geq 2} \gamma^{k-1} t^k = \beta - t + \frac{\gamma t^2}{1-\gamma t}, \quad \text{for } 0 \leq t < \frac{1}{\gamma}, \tag{1.7}$$

has two positive real roots t^* and t^{**} then the Newton-Raphson method (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range defined in (1.4) and it is unique in (1.5).

Theorem 1.5 (**Wang-Zhao α-theory).** *Let us consider $f : D \subseteq \mathbb{R} \to \mathbb{R}$ a continuous and differentiable function in an interval D where the following conditions hold:*

i) $f'(x_0) \neq 0$.

ii) $\beta(x_0, f) = \left|\frac{f(x_0)}{f'(x_0)}\right|$.

iii) $\frac{1}{k!}\left|\frac{f^{(k)}(x_0)}{f'(x_0)}\right| \leq \gamma_k, \quad k \geq 2.$

If the majorizing function

$$\phi_W(t) = \beta - t + \sum_{k\geq 2} \gamma_k t^k. \tag{1.8}$$

has two positive real roots t^* and t^{**} , then method defined in (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$. This solution is located in the range defined in (1.4) and it is unique in (1.5).

2 Main Results

Applying the conditions of the semilocal convergence theorems of Kantorovich (1.1), Gutiérrez (1.3), Smale (1.4) and Wang-Zhao (1.5) to the Kepler's equation, we have obtained, from $E_0 = \pi$, the following results on the existence and uniqueness of solution in terms of the eccentricity e.

Theorem 2.1 (Kantorovich's conditions)**.** *If $e \leq 0.247808$, then Kepler's equation has a solution. This solution is located in the range*

$$[\pi - t^*, \pi + t^*]. \tag{2.1}$$

and it is unique in

$$(\pi - t^{**}, \pi + t^{**}), \tag{2.2}$$

where $\delta = 1/(1+e)$ and

$$\begin{aligned} t^* &= \frac{1-\sqrt{1-\pi e\delta^2}}{e\delta}. \\ t^{**} &= \frac{1+\sqrt{1-\pi e\delta^2}}{e\delta}, \end{aligned}$$

are the roots of the majorizing polynomial

$$\phi_K(t) = \frac{1}{2} e\delta t^2 - t + \pi\delta.$$

In addition, the Newton-Raphson method, defined by

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0, \tag{2.3}$$

starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Kantorovich's theorem to check the following conditions:

i) $f'(E_0) \neq 0$.

ii) $\left|\frac{f(E_0)}{f'(E_0)}\right| \leq \beta$.

iii) $\left|\frac{f'(E)-f'(y)}{f'(E_0)}\right| \leq \gamma(E-y), \quad \forall E,\, y$.

iv) $h = \gamma\beta \leq \frac{1}{2}$.

In this case, we see that

$$\begin{aligned}
f(E) &= E - e\sin E - M \Rightarrow f(E_0) = \pi - M \Rightarrow |f(E_0)| \leq \pi.\\
f'(E) &= 1 - e\cos E \Rightarrow f'(E_0) = 1 + e\\
f'(y) &= 1 - e\cos y.\\
|\frac{f(E_0)}{f'(E_0)}| &\leq \frac{\pi}{1+e} = \beta = \pi\delta.\\
|\frac{f'(E)-f'(y)}{f'(E_0)}| &= \frac{e}{1+e}|\cos E - \cos y| \leq \frac{e}{1+e}|E-y| \Rightarrow \gamma = \frac{e}{1+e} = e\delta
\end{aligned}$$

so the majorizing polynomial is given by (2.3). Then

$$h = \beta\gamma = \pi e\delta^2 \leq \frac{1}{2} \Rightarrow \frac{\pi e}{(1+e)^2} \leq \frac{1}{2} \Rightarrow e \leq 0.247808.$$

The rest of the proof follows immediately from the theorem of Kantorovich. □

Remark. In some cases, it is possible to weaken the Lipschitz condition in Kantorovich's theorem, as it has been indicated in Theorem 1.2. In our case, for the Kepler's equation, $f(x) = x - e\sin x - M$, we have $f'(x) = 1 - e\cos x$.

So, for $x_0 = \pi$, we have

$$\begin{aligned}
\frac{f'(x)-f'(y)}{f'(x_0)} &= \frac{e}{1+e}(\cos y - \cos x)\\
&= \frac{e}{1+e}\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{(2n)!}\Big(y^{2n} - x^{2n}\Big)\\
&= \frac{e}{1+e}\big[\sup_{x,y\in I}\big(\frac{x+y}{2!} - \frac{x^3+x^2y+xy^2+y^3}{4!} + \frac{x^5+x^4y+\cdots+y^5}{6!} - \ldots\big)\big](x-y),
\end{aligned}$$

and

$$L = \frac{e}{1+e}\sup_{x,y\in I}|\frac{x+y}{2!} - \frac{x^3+x^2y+xy^2+y^3}{4!} + \frac{x^5+x^4y+\cdots+y^5}{6!} - \ldots|.$$

For the Lipschitz-center condition:

$$L_0 = \frac{e}{1+e}\sup_{x\in I}|\frac{x+\pi}{2!} - \frac{x^3+x^2\pi+xy^2+\pi^3}{4!} + \frac{x^5+x^4\pi+\cdots+\pi^5}{6!} - \ldots|.$$

So, if $L \leq \gamma = \frac{e}{1+e}$ or $L_0 < L$, then the Theorem 1.2 could be applied, obtaining better bounds for the eccentricity e.

Theorem 2.2 (Gutiérrez's conditions)**.** *If $e \leq 0.129927$ then Kepler's equation has a solution. This solution is located in (2.1) and it is unique in (2.2), where t^* and t^{**} are the positive roots of the majorizing polynomial*

$$\phi_G(t) = \pi\delta - t + \frac{1}{6}e\delta t^3, \tag{2.4}$$

whit $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Gutiérrez's theorem to obtain the following conditions:

i) $\left|\frac{f(E_0)}{f'(E_0)}\right| \leq a.$

ii) $\left|\frac{f''(E_0)}{f'(E_0)}\right| \leq b.$

iii) $\left|\frac{f''(E)-f''(E_0)}{f'(E_0)}\right| \leq K|E-E_0|$

and to calculate the parameters a, b and K. As in the previous result, we have

$$\left|\frac{f(E_0)}{f'(E_0)}\right| \leq \frac{\pi}{1+e} = a. \tag{2.5}$$

$$\tag{2.6}$$

In addition,

$$\begin{aligned} f''(E) &= e\sin E \Rightarrow f''(E_0) = e\sin\pi \Rightarrow f''(E_0) = 0. \\ |f''(E) - f''(E_0)| &= e|\sin E - \sin E_0| \leq e|E-\pi|. \end{aligned}$$

Then

$$\left|\frac{f''(E_0)}{f'(E_0)}\right| \leq 0 = b. \tag{2.7}$$

$$\left|\frac{f''(E)-f''(E_0)}{f'(E_0)}\right| \leq \frac{e}{1+e}|E-E_0| \Rightarrow K = \frac{e}{1+e}. \tag{2.8}$$

Substituting (2.5), (2.7) and (2.8) in (1.6) we obtain (2.4) Let us analyze the polynomial (2.4).

$$\phi'_G(t) = -1 + \frac{1}{2}e\delta t^2.$$

$$\phi'_G(t) = 0 \Leftrightarrow -1 + \frac{1}{2}e\delta t^2 = 0 \Leftrightarrow t = \pm\sqrt{\frac{2}{e\delta}}. \tag{2.9}$$

$$\tag{2.10}$$

Let us consider $\hat{t} = \sqrt{2/(e\delta)}$, then

$$\phi''_G(\hat{t}) = e\delta\sqrt{\frac{2}{e\delta}} > 0$$

and $\phi_G(t)$ has a local minimum at $\hat{t}$. Note that

$$\phi_G(\hat{t}) = \pi\delta - \sqrt{\frac{2}{e\delta}} + \frac{1}{6}e\delta\left(\sqrt{\frac{2}{e\delta}}\right)^3 = \pi\delta - \frac{2\sqrt{2}}{3}\sqrt{\frac{1}{e\delta}} \tag{2.11}$$

So the polynomial $\phi_G(t)$ has positive roots if $\phi_G(\hat{t}) < 0$, that is

$$\pi\delta - \frac{2\sqrt{2}}{3}\sqrt{\frac{1}{e\delta}} < 0 \Leftrightarrow \frac{1}{1+e} < \frac{2}{\sqrt[3]{(3\pi)^2 e}} \Leftrightarrow e \leq 0.129927$$

Accordingly, when $e \leq 0.129927$, Kepler's equation (1.2) has a solution and the rest of the test is a direct application of Theorem 1.3.

□

Theorem 2.3 (Smale's conditions)**.** *If $e \leq 0.142599$, then Kepler's equation has a solution. This solution is located in (2.1) and it is unique in (2.2), where t^* and t^{**} are the positive roots of the majorizing function*

$$\phi_S(t) = \pi\delta - t + \frac{e\delta t^2}{2 - e\delta t}, \tag{2.12}$$

whit $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Smale's α-theorem. So we must find the parameters β, γ and α that appear in the following conditions:

i) $f'(E_0) \neq 0$.

ii) $\left|\frac{f(E_0)}{f'(E_0)}\right| \leq \beta$.

iii) $\frac{1}{k!}\left|\frac{f^{(k)}(E_0)}{f'(E_0)}\right| \leq \gamma^{k-1}, \quad k \geq 2.$

iv) $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$.

As in Theorem 2.1 we have

$$\left|\frac{f(E_0)}{f'(E_0)}\right| \leq \frac{\pi}{1+e} = \beta.$$

Furthermore

$$\begin{array}{lcl} f''(E) = e\sin\pi & \Rightarrow & f''(E_0) = 0. \\ f'''(E) = e\cos\pi & \Rightarrow & |f'''(E_0)| \leq e. \\ & \vdots & \end{array}$$

$$|f^k(E_0)| \leq \begin{cases} 0, & k = 2n, \forall n \in \mathbb{N}; \\ e, & k = 2n+1, \forall n \in \mathbb{N}. \end{cases}$$

Then we must find a constant γ such that

$$\left|\frac{1}{k!}\frac{f^k(E_0)}{f'(E_0)}\right|^{\frac{1}{k-1}} \leq \left[\frac{1}{k!}\frac{e}{1+e}\right]^{\frac{1}{k-1}} \leq \gamma. \tag{2.13}$$

Let us introduce the sequence

$$x_k = \left[\frac{1}{k!}\frac{e}{1+e}\right]^{\frac{1}{k-1}}, \quad k \geq 2.$$

As $e \in [0,1]$, $\{x_k\}$ is a decreasing monotone sequence that converges to 0. So we can choose

$$\gamma = x_2 = \frac{e}{2(1+e)}.$$

Now we must prove

$$\alpha = \beta\gamma \leq \frac{\pi e}{2(1+e)^2} \leq 3 - 2\sqrt{2}. \tag{2.14}$$

This inequality is true for $e \leq 0.142599$, so the hypothesis of Smale's theorem 1.4 are satisfied and the rest of the proof is a consequence of this theorem.

□

Theorem 2.4 (Wang-Zhao's condition). *If $e \leq 0.018826$, then Kepler's equation has a solution. This solution is located in (2.1) and it is unique in (2.2), where t^* and t^{**} are the positive roots of the majorizing polynomial*

$$\phi_{W-Z}(t) = \delta e\ exp(t) - t(1+e\delta) + \delta(\pi - e), \tag{2.15}$$

whit $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. Once again, we take $E_0 = \pi$ in the α-theorem of Wang-Zhao to check the following conditions:

i) $f'(E_0) \neq 0$.

ii) $\left|\frac{f(E_0)}{f'(E_0)}\right| \leq \beta$.

iii) $\frac{1}{k!}\left|\frac{f^{(k)}(E_0)}{f'(E_0)}\right| \leq \gamma_k, \quad k \geq 2$.

We proceed as in the previous theorem to obtain the parameters $\beta = \pi\delta$ and $\gamma_k = e\delta/k!$. So the corresponding majorizing function is

$$\begin{aligned} \phi_{W-Z}(t) = \pi\delta - t + \sum_{k\geq 2}\gamma_k t^k &= \pi\delta - t + \sum_{k\geq 2}\frac{e}{k!}\delta t^k. \\ &= \pi\delta - t + e\delta\sum_{k\geq 2}\frac{t^k}{k!}. \\ &= \delta[e\exp(t) + (\pi - e)] - t(1+e\delta). \end{aligned} \tag{2.16}$$

Let's analyze (2.16)

$$\phi'_{W-Z}(t) = \delta[e\exp(t)] - (1+e\delta). \tag{2.17}$$

So, the critical point of equation (2.17) is attained when

$$\phi'_{W-Z}(\hat{t}) = 0 \Leftrightarrow \hat{t} = \ln\left(1 + \frac{1}{e\delta}\right). \tag{2.18}$$

We check if there is a minimum or maximum. Note that

$$\phi''_{W-Z}(t) = e\delta\exp(t). \tag{2.19}$$

substituting (2.18) en (2.19) it follows that

$$\phi''_{W-Z}(\hat{t}) = e\delta\exp[\ln\left(1 + \frac{1}{e\delta}\right)] = e\delta + 1 > 0,$$

then, there is a minimum at $\hat{t}$ defined by (2.18).

Then the majorizing function $\phi_{W-Z}(t)$ has positive solutions if

$$\phi_{W-Z}(\hat{t}) = (1+\pi\delta) - \ln\left(1 + \frac{1}{e\delta}\right)(1+e\delta) < 0 \tag{2.20}$$

or equivalently, if $e \leq 0.018826$, where 0.018826 is the only positive solution of the equation

$$\exp\left(\frac{1+e+\pi}{1+2e}\right) - \frac{1}{e} - 2 < 0.$$

The rest of the proof follows immediately.

□

3 Numerical and Graphical Experiments for Comparison of Results from the Theorems Proved Above

In the Table 1, the authors compare the increasing functions of the Kantorovich, Gutiérrez, Theorem, $\alpha-$theory of Smale and the $\alpha-$theory of Wang-Zhao, applied to the Kepler equation, depending on the points $E_0 = M$ and $E_0 = \pi$, and their respective parameters δ.

Table 1. Majorizing functions for different theories of semilocal convergence according to the starting points and the parameter δ

Theory	$E_0 = M$, $\delta = \frac{e}{1-e}$	$E_0 = \pi$, $\delta = \frac{1}{1+e}$
Kantorovich	$\phi_K(t) = \frac{1}{2}\delta t^2 - t + \delta$	$\phi_K(t) = \frac{1}{2}e\delta t^2 - t + \pi\delta$
Gutiérrez	$\phi_G(t) = \delta - t + \frac{1}{2}\delta t^2 + \frac{1}{6}\delta t^3$	$\phi_G(t) = \pi\delta - t + \frac{e\delta\, t^3}{6}$
Smale's α-theory	$\phi_S(t) = \delta - t + \frac{\sqrt{\delta}\, t^2}{\sqrt{6}-\sqrt{\delta}\, t}$	$\phi_S(t) = \pi\delta - t + \frac{e\delta\, t^2}{2-e\delta\, t}$
Wang-Zhao's α-theory	$\phi_{WZ}(t) = \delta \exp(t) - (1+\delta)t$	$\phi_{WZ}(t) = e\delta \exp(t) - (1+e\delta)t + \delta(\pi - e)$

In the Table 2, the convergence conditions of the theorems mentioned above, are compared depending of the starting points, of the parameters δ and the maximum value of e theorems for which guarantee convergence of the Newton Method to a solution of the Kepler equation.

Table 2. Values of δ and e that guarantee the existence of solution for Kepler's equation by applying different semilocal convergence techniques for Newton's method, according to the starting points

Theory	$E_0 = M$	$E_0 = \pi$	$E_0 = M$	$E_0 = \pi$
Kantorovich	$e \leq 0.4142$	$e \leq 0.2478$	$\delta = 0.707107$	$\delta = 0.801416$
Gutiérrez	$e \leq 0.3759$	$e \leq 0.1299$	$\delta = 0.6023$	$\delta = 0.885074$
Smale's α-theory	$e \leq 0.3594$	$e \leq 0.142599$	$\delta = 0.561069$	$\delta = 0.875198$
Wang-Zhao's α-theory	$e \leq 0.3678$	$e \leq 0.018826$	$\delta = 0.581977$	$\delta = 0.981527$

In Tables 3 and 4 we have calculated the values t^* and t^{**} for real situations corresponding to the planets Venus, Earth and Neptuno, as well as satellites Amalthea, Charon, Dione, Enceladus and Ganymede. We show the corresponding radius of existence and uniqueness of solution to the Kepler equation. We have denoted these values t^* and t^{**} with a subscript that refers to the theory with which this related: K for Kantorovich, G for strong Kantorovich, S for Smale and WZ for Wang-Zhao.

Table 3. Radius of existence for four majorizing functions applied to real cases

Cases	t^*_K	t^*_G	t^*_S	t^*_{WZ}
Neptune ($e = 0.0086$)	3.1573	3.15963	3.1579	3.31201
Earth($e = 0.0167$)	3.17324	3.17855	3.175	3.66213
Venus ($e = 0.0068$)	3.15397	3.15575	3.15434	3.26909
Amalthea Satellite ($e = 0.003$)	3.14701	3.14774	3.14708	3.19248
Charon Satellite ($e = 0.0035$)	3.14792	3.14878	3.14801	3.20169
Dione Satellite ($e = 0.002$)	3.14519	3.14568	3.14523	3.17473
Enceladus Satellite ($e = 0.004$)	3.14883	3.14983	3.14895	3.21113
Ganymede Satellite ($e = 0.001$)	3.14339	3.14363	3.1434	3.1578

Table 4. Uniqueness radius for the four majorizing functions

Cases	t_K^{**}	t_G^{**}	t_S^{**}	t_{WZ}^{**}
Neptune ($e = 0.0086$)	231.401	24.8055	115.679	3.31201
Earth ($e = 0.0167$)	116.474	17.1554	59.2502	4.49618
Venus ($e = 0.0068$)	292.964	28.1018	146.465	6.10804
Amalthea Satellite ($e = 0.003$)	665.52	43.1315	332.752	7.22812
Charon Satellite ($e = 0.0035$)	570.281	39.8122	285.132	7.02524
Dione Satellite ($e = 0.002$)	998.855	53.1864	499.422	7.74952
Enseladus Satellite ($e = 0.004$)	498.851	37.1363	249.416	6.84699
Ganymede Satellite ($e = 0.001$)	1998.86	75.8787	999.426	8.61008

Finally, in the Table 5, we see the majoring functions and the real roots occur when applied theories cited under optimal conditions and actual Orthosie, Themisto, Europa, 1994 JR_1, and 2000 PV_{29}, satellites and Thebe, Metis and Adrastea besides asteroids. In addiction, for each of the tables, they have built their respective graphs. The red color corresponds to Gutiérrez's theory, the blue to Smale's theory, the green to the Wang-Zhao's theory and the gray to Kantorovich's theory.

Table 5. majorizing functions and radius of convergence of the different theories applied under optimal conditions

Theory	Cases	Mayority function	t^*	t^{**}
Kantorovich	Orthosie ($e = 0.243$)	$\phi_K(t) = 0.098\,t^2 - t + 2.527$	4.573	5.648
Kantorovich	Themisto ($e = 0.212$)	$\phi_K(t) = 0.087\,t^2 - t + 2.593$	3.967	7.490
Gutiérrez	Europa ($e = 0.096$)	$\phi_G(t) = 2.867 - t + 0.015\,t^3$	3.485	5.965
Gutiérrez	1994 JR_1 ($e = 0.12$)	$\phi_G(t) = 2.805 - t + 0.018\,t^3$	3.737	4.879
Wang-Zhao	2000 PV_{29} ($e = 0.013$)	$\phi_W(t) = 0.013\exp(t) - 1.013\,t + 3.088$	3.447	5.074
Wang-Zhao	Thebe ($e = 0.018$)	$\phi_{WZ}(t) = 0.017\exp(t) - 1.017\,t + 3.070$	3.707	4.411
Smale	Metis ($e = 0.00002$)	$\phi_S(t) = 3.1 - t + \frac{0.00002\,t^2}{2-0.00002\,t}$	3.14	49999.4
Smale	Adrastea ($e = 0.0015$)	$\phi_S(t) = 3.1 - t + \frac{0.0015\,t^2}{2-0.0015\,t}$	3.144	666.1

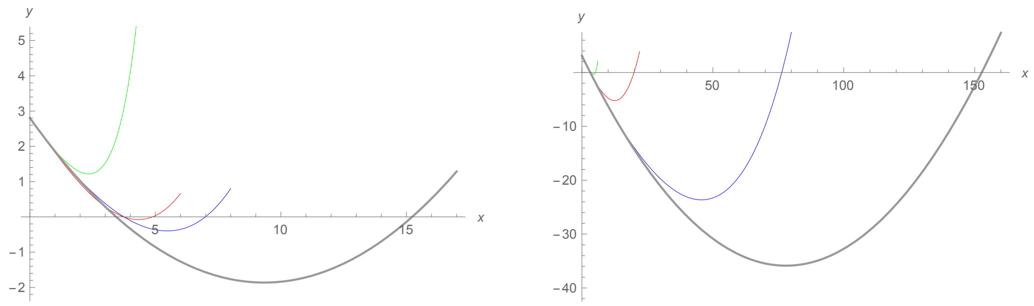

Fig. 1. Details of the graphs of the majorizing functions $\phi_S(t)$ (blue), $\phi_W(t)$ (green), $\phi_G(t)$ (red) and $\phi_K(t)$ (gray), for cases of the 1994 JR_1 satellite, ($e = 0.12$), (left) and 2000 PV_{29}, satellite, ($e = 0.013$), (right)

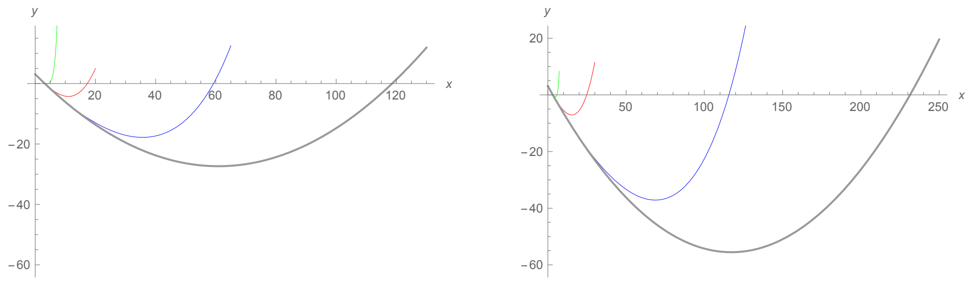

Fig. 2. Details of the graphs of the majorizing functions $\phi_S(t)$ (blue), $\phi_W(t)$ (green), $\phi_G(t)$ (red) and $\phi_K(t)$ (gray), for cases of the planet Earth, $(e = 0.0167)$, (left) and planet Neptune, $(e = 0.0086)$, (right)

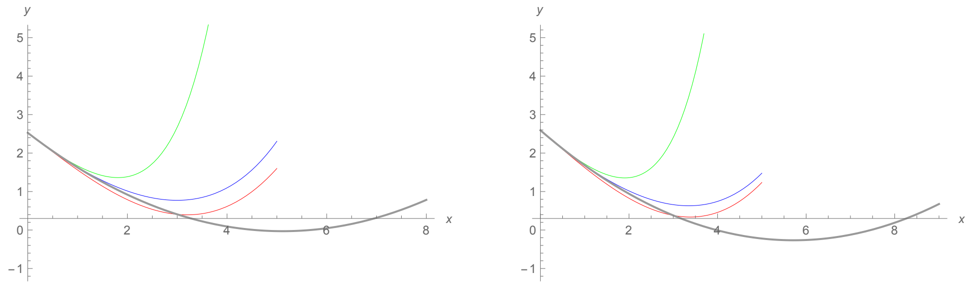

Fig. 3. Details of the graphs of the majorizing functions $\phi_S(t)$ (blue), $\phi_W(t)$ (green), $\phi_G(t)$ (red) and $\phi_K(t)$ (gray), for cases of the Orthosie satellite, $(e = 0.243)$, (left) and Themisto satellite, $(e = 0.212)$, (rigth)

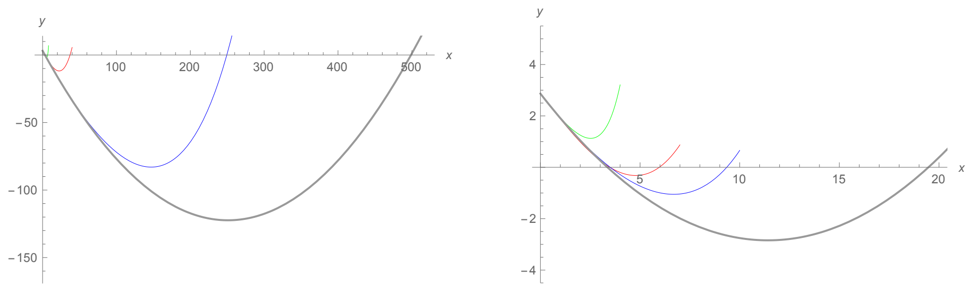

Fig. 4. Details of the graphs of the majorizing functions $\phi_S(t)$ (blue), $\phi_W(t)$ (green), $\phi_G(t)$ (red) and $\phi_K(t)$ (gray), for cases of the Enceladus satellite, $(e = 0.004)$, (left) and Europa satellite, $(e = 0.096)$, (rigth)

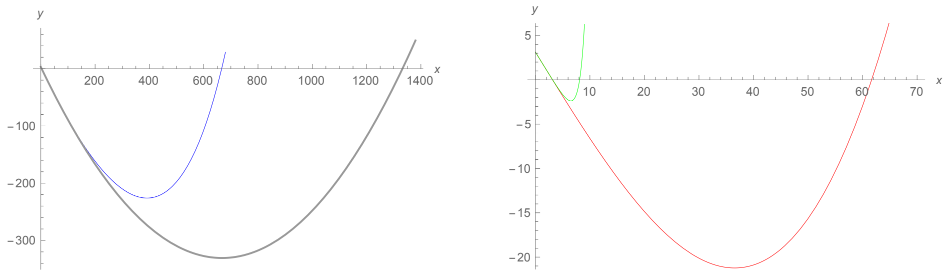

Fig. 5. Details of the graphs of the majorizing functions $\phi_K(t)$ (gray) and $\phi_W(t)$ (green) for cases of the Adrastea satellite, $(e = 0.0015)$, (left), $\phi_G(t)$ (red) and $\phi_W(t)$ (green) (right)

4 Conclusion

Given all the results obtained in the previous sections, both the starting point $E_0 = M$ and $E_0 = \pi$, it is concluded that Kepler's equation (1.1), the best results are deduced from the Kantorovich theory. In fact, regarding the values of the eccentricity e, Kantorovich theorem gives the highest value among the considered theorems. For more detail, in the Table 2 the different theories discussed in this article are shown, and the corresponding values for the parameter δ and eccentricity e. In addition, it is found that for the starting point $E_0 = \pi$, eccentricity values that ensure convergence of Newton's method applied to the equation of Kepler, become more restrictive. Draws attention the case Wang-Zhao $\alpha-$theory, for which he won a $e \leq 0.018826$. This restriction caused that actual cases of 1994 JR_1 satellite and Europa satellite, no convergence is obtained, see Figs. 1 and 4.

Under the optimum conditions for real cases of Orthosie, Themiso, Europe satellites and for Thebe, Metis and Adrastea asteroids, polynomials of the various theories presented convergence conditions being Kantorovich which presented the best results again. Other theories expressed similar behavior, refer to Table 5 and Fig. 2.

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Competing Interests

Authors have declared that no competing interests exist.

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