Asian Research Journal of Mathematics

16(10): 56-71, 2020; Article no.ARJOM.62190

ISSN: 2456-477X

Variational and Topological Methods for a Class of Nonlinear Equations which Involves a Duality Mapping

Jenica Cringanu[1*]

[1] *Department of Mathematics, University of Galati, Galati 800008, Romania.*

Authors contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i1030230
Editor(s):
(1) Dr. Xingting Wang, Howard University, USA.
Reviewers:
(1) Jalil Manafian, University of Tabriz, Iran.
(2) Amruta Atul Bhandari, R. C. Patel Institute of Technology, India.
Complete Peer review History: http://www.sdiarticle4.com/review-history/62190

Received: 24 August 2020
Accepted: 30 October 2020
Published: 06 November 2020

Original Research Article

Abstract

The purpose of this paper is to show the existence results for the following abstract equation

$$J_p u = N_f u,$$

where J_p is the duality application on a real reflexive and smooth X Banach space, that corresponds to the gauge function $\varphi(t) = t^{p-1}$, $1 < p < \infty$. We assume that X is compactly imbedded in $L^q(\Omega)$, where Ω is a bounded domain in $R^N, N \geq 2$, $1 < q < p^*$, p^* is the Sobolev conjugate exponent.
$N_f : L^q(\Omega) \to L^{q'}(\Omega), \frac{1}{q} + \frac{1}{q'} = 1$, is the Nemytskii operator that Caratheodory function generated by a $f : \Omega \times R \to R$ which satisfies some growth conditions. We use topological methods (via Leray-Schauder degree), critical points methods (the Mountain Pass theorem) and a direct variational method to prove the existence of the solutions for the equation $J_p u = N_f u$.

Keywords: Duality mapping; leray-schauder degree; mountain pass theorem, p-Laplacian.

AMS Subject Classification: 58C15, 35J20, 35J60, 35J65.

*Corresponding author: E-mail: jcringanu@ugal.ro;
ORCID ID: https://orcid.org/0000-0002-1809-3167*

1 Introduction

The subject of this paper is the existence of the solutions for the abstract equation

$$J_p u = N_f u, \tag{1.1}$$

in the following functional framework:

(H_1) $1 < p < \infty$;

(H_2) X is a real smooth and reflexive Banach space, compactly imbedded in $L^q(\Omega)$, where $\Omega \subset R^N, N \geq 2$, is a bounded domain with smooth boundary and

$$1 \; < q < p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad N > p \\ +\infty & \text{if} \quad N \leq p \end{cases} \quad ;$$

(H_3) For any gauge function $\varphi : R_+ \to R_+$, the corresponding duality mapping $J_\varphi : X \to X^*$ is continuous and satisfies the (S_+) condition: if $x_n \rightharpoonup x$ (weakly) in X and $\limsup\limits_{n\to\infty} \langle J_\varphi x_n, x_n - x\rangle \leq 0$ then $x_n \to x$ (strongly) in X;

(H_4) $J_p : X \to X^*$ is the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}, t \geq 0$;

(H_5) $N_f : L^q(\Omega) \to L^{q'}(\Omega)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, defined by $(N_f u)(x) = f(x, u(x))$, for $u \in L^q(\Omega)$, $x \in \Omega$, is the Nemytskii operator generated by the Caratheodory function $f : \Omega \times R \to R$, which satisfies the growth condition

$$|f(x,s)| \leq c|s|^{q-1} + b(x), \text{ for } x \in \Omega, s \in R, \tag{1.2}$$

where $c > 0$ is constant and $b \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

We establish the fact that in the case of a Caratheodory function, the assertion "$x \in \Omega$" must be understood as "$a.e.x \in \Omega$".

In order to show the existence of the problem's (1.1) solutions, we use topological methods(via Leray-Schauder degree), critical points methods(the Mounain Pass theorem) and a direct variational method. The growth condition (1.2) is the one adoptated by Dinca, Jebelean and Mawhin [1] in the study of the existence of the solutions for the following Dirichlet problem:

$$\begin{aligned} -\Delta_p u &= f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.3}$$

By the solution of the problem (1.3) we mean an element $u \in W_0^{1,p}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_\Omega f(x,u) v \, dx \quad \text{for all} \quad v \in W_0^{1,p}(\Omega). \tag{1.4}$$

On the other hand, if $W_0^{1,p}(\Omega)$ is defined with the norm

$$\| u \|_{1,p} = \| \, |\nabla u| \, \|_{0,p} \, ,$$

the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$ is exactly $-\Delta_p$:

$$J_p = -\Delta_p : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)^* = W^{-1,p'}(\Omega), \; \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega),$$

(see e.g.[1] or [2]). Let $i : W_0^{1,p}(\Omega) \to L^q(\Omega)$ be the compact imbedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ and $i' : L^{q'}(\Omega) \to W^{-1,p'}(\Omega)$ be its adjoint. If $N_f : L^q(\Omega) \to L^{q'}(\Omega)$ is the Nemytskii operator generated by f, it is easy to check that $u \in W_0^{1,p}(\Omega)$ is a solution of the problem (1.3) in the sens of (1.4) if and only if u is a solution of the operator equation

$$J_p u = \left(i' N_f i\right) u.$$

This remark was the departure point in considering the abstract equation (1.1) in the hypotheses (H_1) - (H_5). Returning to the equation (1.1), by its solution we mean an element $u \in X$ which satisfies

$$J_p u = (i' N_f i) u, \tag{1.5}$$

where i is the compact imbedding of X into $L^q(\Omega)$ and $i' : L^{q'}(\Omega) \to X^*$ is its adjoint.

Let us remark that according to (1.2), N_f is well-defined, continuous and bounded from $L^q(\Omega)$ into $L^{q'}(\Omega)$, such that $i' N_f i$ is also well-defined and compact from X into X^* (see e.g.[1]). Also, the functional $\psi : L^q(\Omega) \to R$, defined by $\psi(u) = \int_\Omega F(x,u)dx$, is C^1 on $L^q(\Omega)$ and then on X and $\psi'(u) = N_f u$ for all $u \in X$ (see e.g.[3]), and notice that (1.5) is equivalent with

$$\langle J_p u, v\rangle = \langle N_f(iu), iv\rangle_{L^{q'}(\Omega), L^q(\Omega)} = \int_\Omega f(x,u) v \, dx \quad \text{for all} \quad v \in X. \tag{1.6}$$

The abstract model reprezented by (1.1) allows us to find, in a unitary manner, the existence results given in [1] for the Dirichlet problem (1.3) and also for the following Neumann problem:

$$-\Delta_p u + |\, u\,|^{p-2} u \quad = \quad f(x,u) \quad \text{in} \quad \Omega, \tag{1.7}$$

$$|\, \nabla u\,|^{p-2} \frac{\partial u}{\partial n} \quad = \quad 0 \quad \text{on} \quad \partial\Omega, \tag{1.8}$$

(see[4]).

2 Preliminary Results

2.1 Duality mappings

Let $(X, \|\cdot\|)$ be a real Banach space, X^* its dual and $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. The norm on X^* is denonted by $\|\cdot\|_*$. A continuous function $\varphi : R_+ \to R_+$ is called a gauge function if it is strictly increasing, $\varphi(0) = 0$, and $\varphi(r) \to \infty$ with $r \to \infty$. By duality mapping corresponding to the gauge function φ we mean the operator $J_\varphi : X \to \mathcal{P}(X^*)$ defined by

$$J_\varphi x = \{x^* \in X^* : \quad \langle x^*, x\rangle = \varphi(\|\, x\,\|)\,\|\, x\,\|, \quad \|\, x^*\,\| = \varphi(\|\, x\,\|)\}, \quad \text{for} \quad x \in X.$$

By the Hahn-Banach theorem we have

$$J_\varphi x \neq \emptyset \quad \text{for all} \quad x \in X.$$

The following theorem contains some of the most important properties of the duality mapping as in:

Theorem 2.1. *If φ is a gauge function, therefore: (i) for every $x \in X$, $J_\varphi x$ is a bounded, closed and convex subset of X^*; (ii) J_φ is monotone:*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq (\varphi(\|\, x_1\,\|) - \varphi(\|\, x_2\,\|))\,(\|\, x_1\,\| - \|\, x_2\,\|) \geq 0$$
$$\textit{for each} \quad x_1, x_2 \in X \quad \textit{and} \quad x_1^* \in J_\varphi x_1, \quad x_2^* \in J_\varphi x_2;$$

(iii) for every $x \in X$, $J_\varphi x = \partial\Phi(x)$, where $\Phi(x) = \int_o^{\| x \|} \varphi(t)dt$ and $\partial\Phi : X \to \mathcal{P}(X^*)$ is the subdifferential of Φ in the sense of convex analysis, i.e.

$$\partial\Phi(x) = \{x^* \in X^* : \ \Phi(y) - \Phi(x) \geq \langle x^*, y - x\rangle, \ \text{for all } y \in X\}.$$

For the proof see Browder [5], Cioranescu [6] or Lions [2].

Remark 2.1. *We notice that a functional $f : X \to R$ is said to be Gateaux differentiable at $x \in X$ if there exists $f'(x) \in X^*$ such that*

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = \left\langle f'(x), h\right\rangle \quad \text{for all} \quad h \in X.$$

If the convex function $f : X \to R$ is Gâteaux differentiable in $x \in X$ then it verifies that $\partial f(x) = \{f'(x)\}$.

In the following propositions we recall other properties of the duality mapping:

Proposition 2.1. *$J_\varphi : X \to \mathcal{P}(X^*)$ is single valued $\Leftrightarrow$ X is smooth $\Leftrightarrow$ the norm of X is Gateaux differentiable on $X \setminus \{0\}$.*

Proposition 2.2. *If X is reflexive and $J_\varphi : X \to X^*$, then J_φ is demicontinuous : if $x_n \to x$ (strongly) in X, then $J_\varphi x_n \rightharpoonup J_\varphi x$ (weakly) in X^*.*

Notice that X has the Kadeč-Klee property ($(K-K)$ in short terms) if it is strictly convex and for any sequence $(x_n) \subset X$ such that $x_n \rightharpoonup x$ (weakly) in X and $\| x_n \| \to \| x \|$ it follows that $x_n \to x$ (strongly) in X.

Proposition 2.3. *If X has the $(K-K)$ property and J_φ is single valued then J_φ satisfies the (S_+) condition.*

We remark that if X is locally uniformly convex then X has the $(K-K)$ property and then, if in addition, J_φ is single valued, we obtain that J_φ satisfies the (S_+) condition. At the same time, if X is reflexive and X^* has the $(K-K)$ property then $J_\varphi : X \to X^*$ is continuous.

Proposition 2.4. *J_φ is single valued and continuous if and only if the norm of X is Fréchet differentiable.*

Proposition 2.5. *If X is reflexive and $J_\varphi : X \to X^*$ then J_φ is surjective. If, in addition X is locally uniformly convex then J_φ is bijective, with its inverse J_φ^{-1} bounded, continuous and monotone.*

For the details and the proofs see e.g. [5], [6] or [1]. The propositions 2.3 and 2.4 offer sufficient conditions to satisfy hypothesis (H_3). Furthermore, $(X, \| \ \|_X)$ is a reflexive real Banach space, compactly imbedded in the real Banach space $(Z, \| \ \|_Z)$, the continuity of the imbedding i being given by

$$\| iv \|_Z \leq c_Z \| v \|_X \quad \text{for all} \quad v \in X. \tag{2.1}$$

Using these hypotheses we denote

$$\begin{aligned} \lambda_1 &= \inf\left\{ \frac{\| v \|_X^p}{\| iv \|_Z^p} : \quad v \in X \setminus \{0\} \right\} = \\ &= \inf\left\{ \| v \|_X^p : \quad v \in X, \| iv \|_Z = 1 \right\}. \end{aligned} \tag{2.2}$$

Then λ_1 is attained (see e.g.[7]) and $\lambda_1^{-\frac{1}{p}}$ is the best constant c_Z for the imbedding of X into Z (inequality (2.1)). Using the following theorem we can emphasize another meaning of λ_1: if X

and Z have both G-differentiable norm, then λ_1 is the first eigenvalue of the pair of the duality mapping. So, from the hypotheses on X and Z written at the begining of this section, we denote by $J_{p,XX^*}: X \to X^*$ and $J_{p,ZZ^*}: Z \to Z^*$ the duality mappings (assumed to be single-valued) on X, respectively on Z, corresponding to the same gauge function $\varphi(t) = t^{p-1}$. We say that $\lambda \in R$ is an eigenvalue of the pair (J_{p,XX^*}, J_{p,ZZ^*}) if there exists
$u \in X \setminus \{0\}$ such that

$$J_{p,XX^*}u = \lambda J_{p,ZZ^*}u. \tag{2.3}$$

Equality (2.3) is meant in the sense

$$\langle J_{p,XX^*}u, v\rangle_{X^*,X} = \lambda \langle J_{p,ZZ^*}(iu), iv\rangle_{Z^*,Z} \quad \text{for all } v \in X.$$

The function u in (2.3) is called eigenfunction of the pair (J_{p,XX^*}, J_{p,ZZ^*}) corresponding to the eigenvalue λ.

Theorem 2.2. (see [7], Theorem 4) *If both of X and Z are with G-differentiable norm, then:*
(i) If λ is an eigenvalue of the pair (J_{p,XX^*}, J_{p,ZZ^*}) then $\lambda \geq \lambda_1$; (ii) λ_1 is an eigenvalue of the pair (J_{p,XX^*}, J_{p,ZZ^*}).

Let us remark that, if $Z = (L^p(\Omega), \|\cdot\|_{0,p})$, then the duality mapping
$J_{p,ZZ^*}: L^p(\Omega) \to L^{p'}(\Omega)$ is defined by

$$\langle J_{p,ZZ^*}u, v\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \int_\Omega |\, u\,|^{p-2}\, uvdx \quad \text{for all } u, v \in L^p(\Omega).$$

So, if $Z = (L^p(\Omega), \|\cdot\|_{0,p})$, then $\lambda \in R$ is an eigenvalue of the pair (J_{p,XX^*}, J_{p,ZZ^*}) (for short we say that λ is an eigenvalue for J_p) if there exists a certain $u \in X \setminus \{0\}$ such that

$$\langle J_p u, v\rangle_{X^*,X} = \lambda \int_\Omega |\, u\,|^{p-2}\, uvdx \quad \text{for all} \quad v \in X.$$

Moreover,

$$\lambda_1 = \inf_{\substack{v\in X\\ v\neq o}} \frac{\|\, v\,\|_X^p}{\|\, v\,\|_{0,p}^p} > 0 \tag{2.4}$$

is attained and it is the smallest eigenvalue for J_p. By theorem 2.1 we have

$$J_p u = \partial\Phi(u) \;\text{ for all } u \in X, \text{ where}$$

$$\Phi(u) = \int_0^{\|\,u\,\|} \varphi(t)dt = \frac{1}{p}\,\|\, u\,\|^p \quad \text{for all } u \in X.$$

Since Φ is convex and X is smooth, it results that Φ is Gateaux differentiable on X and

$$\partial\Phi(u) = \{\Phi'(u)\} \quad \text{for all } u \in X.$$

So, $J_p u = \Phi'(u)$ for all $u \in X$ and from the continuity of J_p (see(H_3)) it results that $\Phi \in C^1(X, R)$. Moreover, $\langle J_p u, u\rangle = \varphi(\|\, u\,\|)\,\|\, u\,\| = \|\, u\,\|^p$ for all $u \in X$. Therefore, the functional $\mathcal{F}: X \to R$ defined by

$$\mathcal{F}(u) = \Phi(u) - \Psi(u) = \frac{1}{p}\,\|\, u\,\|^p - \int_\Omega F(x,u)dx \;\text{ for all } u \in X, \tag{2.5}$$

is C^1 on X and

$$\mathcal{F}'(u) = \Phi'(u) - \Psi'(u) = J_p u - N_f u \;\text{ for all } u \in X.$$

Then $u \in X$ is a solution of the equation (1.1) if and only if u is a critical point for $\mathcal{F}$, i.e.

$$\mathcal{F}'(u) = 0.$$

2.2 Abstract existence results

In the following section we give some results that will be used furthermore.

Theorem 2.3. *(Mountain Pass Theorem) Let X be a real Banach space and $I \in C^1(X, R)$, satisfaying the Palais-Smale(PS) condition. Suppose $I(0) = 0$ and (i) there are constants $\rho > 0, \alpha > 0$ such that $I_{\big|\| x \|=\rho} \geq \alpha$; (ii) there is an element $e \in X$, $\| e \| > \rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized by

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u),$$

where $\Gamma = \{g \in C([0, 1]), X) : g(0) = 0, g(1) = e\}$.

Let us remark that each critical point u at level c $(I'(u) = 0, I(u) = c)$ is a nontrivial one. Notice that the functional $I \in C^1(X, R)$ satisfies the (PS) condition if every sequence $(u_n) \subset X$ for which $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. The second result is a surjectivity Fredholm-type result obtained by Dinca [8].

Theorem 2.4. *Let X and Y be a real Banach spaces and $T, S : X \to Y$ two (generaly nonlinear) operator such that*
(i) T is bijective and T^{-1} is continuous;
(ii) S is compact. Let $\lambda \neq 0$ be a real number such that
(iii) $\| (\lambda T - S)x \| \to \infty$ as $\| x \| \to \infty$;
(iv) there is a constant $R > 0$ such that (iv_1) $\| (\lambda T - S)x \| \to \infty$ as $\| x \| \to \infty$; (iv_2) $d_{LS}(I - T^{-1}(\frac{1}{\lambda}S), B_R, 0) \neq 0$. Then $\lambda T - S$ is surjective from X into Y.

Remark 2.2. *A sufficient condition to satisfy hypothesis (iv) in theorem 2.4 is, at the begining, to exist a constant $r > 0$ such that, from $x = tT^{-1}(\frac{1}{\lambda}S)$, with $x \in X$ and $t \in [0, 1]$, it follows that $\| x \| < r$(see [8]).*

Now we give a multiple version of the "Mountain Pass Theorem".

Theorem 2.5. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, R)$ be even, satisfying (PS) condition, $I(0) = 0$, and: (i) there are constants $\rho > 0, \alpha > 0$ such that $I_{\big|\| x \|=\rho} \geq \alpha$; (ii) for each finite dimensional subspace X_1 of X, the set $\{x \in X_1 : I(x) \geq 0\}$ is bounded. Then I possesses an unbounded sequence of critical values.*

For the proof and the details see e.g. Ambrosetti and Rabinowitz[9], Cringanu and Dinca[10], Kavian[11], Mawhin and Willem[12] or Rabinowitz[13].

3 Existence Results for the Problem (1.1)

3.1 Existence of solution using a Leray-Schauder technique

Since $X \to L^q(\Omega)$ is compact, the diagram

$$X \xrightarrow{i} L^q(\Omega) \xrightarrow{N_f} L^{q'}(\Omega) \xrightarrow{i'} X^*$$

shows that N_f (by which we mean $i' N_f i$) is compact. From the proposition 2.5 it results that the operator $J_p : X \to X^*$ is bijective with its inverse J_p^{-1} bounded continuous, and monotone. Conseqently (1.1) can be written as:

$$u = (J_p^{-1})N_f u,$$

with $J_p^{-1}N_f : X \to X$ a compact operator. Let be the operator $T = J_p^{-1}N_f$ and using an "a priori estimate method" we are going to prove that the compact operator T has at least one fixed point (see Dinca and Jebelean[7]). For this it is sufficient to show that the set

$$S = \{u \in X : u = \alpha T u \text{ for some } \alpha \in [0,1]\}$$

is bounded in X. By (1.2), for $u \in X$ we have

$$\| Tu \|^p = \langle J_p(Tu), Tu\rangle = \langle N_f u, Tu\rangle =$$
$$= \int_\Omega f(x,u)Tu\, dx \le \int_\Omega (c\,|\,u\,|^{q-1} + |\,b\,|)\,|\,Tu\,|\, dx.$$

If $u \in S$, that is $u = \alpha T u$ with $\alpha \in [0,1]$, we have

$$\begin{aligned}\| Tu \|^p &\le \int_\Omega (c\,|\,u\,|^{q-1} + |\,b\,|)\,|\,Tu\,|\,dx = \int_\Omega (c\alpha^{q-1}\,|\,Tu\,|^{q-1} + |\,b\,|)\,|\,Tu\,|\,dx \le \\ &\le c\alpha^{q-1} \| Tu \|_{0,q}^q + \| b \|_{0,q'} \| Tu \|_{0,q} \le c \| Tu \|_{0,q}^q + \| b \|_{0,q'} \| Tu \|_{0,q} \le \\ &\le cc_1^q \| Tu \|^q + c_1 \| b \|_{0,q'} \| Tu \|,\end{aligned}$$

the constant $c_1 > 0$ corresponding to the continuous imbedding $X \to L^q(\Omega)$. Thererfore, for every $u \in S$ it results

$$\| Tu \|^p - c_2 \| Tu \|^q - c_3 \| Tu \| \le 0,$$

with $c_2, c_3 > 0$ constants. Let us remark that, if $q \in (1,p)$ then by the above inequality there is a constant $a > 0$ such that $\| Tu \| \le a$ for $u \in S$ and then

$$\| u \| = \alpha \| Tu \| \le \alpha a \le a, \text{ for } u \in S,$$

that means S is bounded. We have obtained:

Theorem 3.1. *Assume that X is locally uniformly convex, $J_p : X \to X^*$ and the Caratheodory function f satisfies (1.2) with $q \in (1,p)$. Then the operator $J_p^{-1}N_f$ has one fixed point in X and consequently the problem (1.1) has solution. In addition, the set of solutions of the problem (1.1) is bounded in X.*

3.2 Existence of solution using the Mountain Pass Theorem

We assume that the Caratheodory function f satisfies the growth condition (1.2). Then F is a Caratheodory function and there exists $c_1 > 0$, constant and a function $c \in L^1(\Omega)$, $c \ge 0$ such that

$$|\,F(x,s)\,| \le c_1\,|\,s\,|^q + c(x) \text{ for } x \in \Omega, s \in R. \tag{3.1}$$

The preliminary results are:

Proposition 3.1. *If $(u_n) \subset X$ is bounded and $\mathcal{F}'(u_n) \to 0$ as $n \to \infty$, then (u_n) has a convergent subsequence.*

Proof. Since X is reflexive, passing to a subsequence, if necessary, we may assume that $u_n \rightharpoonup u$ (weakly) in X. By $\mathcal{F}'(u_n) \to 0$, $u_n - u \rightharpoonup 0$ we obtain $\left\langle \mathcal{F}'(u_n), u_n - u \right\rangle \to 0$, or equivalently

$$\langle J_p u_n, u_n - u\rangle - \langle N_f u_n, u_n - u\rangle \to 0.$$

Since $u_n \rightharpoonup u$ in X, by the compact imbedding $X \to L^q(\Omega)$, it results that $u_n \to u$ in $L^q(\Omega)$ and then $\langle N_f u_n, u_n - u\rangle \to 0$, because

$$|\,\langle N_f u_n, u_n - u\rangle\,| \le \| N_f u_n \|_{0,q'} \| u_n - u \|_{0,q},$$

and $(N_f u_n)$ is bounded in $L^{q'}(\Omega)$. Consequently, $\langle J_p u_n, u_n - u\rangle \to 0$ and by the (S_+) condition of J_p it results that $u_n \to u$ in X. □

Theorem 3.2. *Assume that there exist $\theta > p$ and $s_0 > 0$ such that*

$$\theta F(x,s) \leq sf(x,s) \ \ \textit{for} \ \ x \in \Omega, \ |\, s \,| \geq s_0. \tag{3.2}$$

Then $\mathcal{F}$ satisfies the (PS) condition.

Proof. According to the proposition 3.1 it is enough to prove that any sequence $(u_n) \subset X$ for which $(\mathcal{F}(u_n))$ is bounded and $\mathcal{F}^{'}(u_n) \to 0$, is also bounded. Let $d \in R$ be such that $\mathcal{F}(u_n) \leq d$ for all $n \in N$. For every $n \in N$ we define

$$\Omega_n = \{x \in \Omega : |\, u_n(x) \,| \geq s_0\}\,,\ \Omega_n^{'} = \Omega \setminus \Omega_n.$$

Therefore

$$\mathcal{F}(u_n) = \frac{1}{p} \parallel u_n \parallel^p - \left(\int_{\Omega_n} F(x,u_n)dx + \int_{\Omega_n'} F(x,u_n)dx \right) \leq d. \tag{3.3}$$

If $x \in {\Omega_n}'$ then $|\, u_n(x) \,| < s_0$ and by (3.1) it holds

$$|\, F(x,u_n) \,| \leq c_1 \,|\, u_n(x) \,|^q + c(x) \leq c_1 s_0^q + c(x),$$

and hence

$$\int_{\Omega_n'} F(x,u_n)dx \leq c_1 s_0^q meas(\Omega) + \int_{\Omega} c(x)dx = k_1. \tag{3.4}$$

If $x \in \Omega_n$ then $|\, u_n(x) \,| \geq s_0$ and by (3.2) it follows

$$F(x,u_n) \leq \frac{1}{\theta} f(x,u_n)u_n,$$

which gives

$$\int_{\Omega_n} F(x,u_n)dx \leq \int_{\Omega_n} \frac{1}{\theta} f(x,u_n)u_n dx = \frac{1}{\theta} \left(\int_{\Omega} f(x,u_n)u_n dx - \int_{\Omega_n'} f(x,u_n)u_n dx \right). \tag{3.5}$$

By the growth condition (1.2) we have

$$\left| \int_{\Omega_n'} f(x,u_n)u_n dx \right| \leq \int_{\Omega_n'} (c \,|\, u_n \,|^q + |\, b(x) \,|\, |\, u_n \,|)\, dx \leq c s_0^q meas(\Omega) + s_0 \int_{\Omega} |\, b(x) \,| dx = k_2,$$

which yields

$$-\frac{1}{\theta} \int_{\Omega_n'} f(x,u_n)u_n dx \leq \frac{k_2}{\theta}. \tag{3.6}$$

Finally, by (3.3), (3.4), (3.5) and (3.6) we obtain

$$\frac{1}{p} \parallel u_n \parallel^p - \frac{1}{\theta} \int_{\Omega} f(x,u_n)u_n dx \leq d + k_1 + \frac{k_2}{\theta} = k. \tag{3.7}$$

Since $\mathcal{F}^{'}(u_n) \to 0$ as $n \to \infty$ there is $n_0 \in N$ such that

$$\left| \left\langle \mathcal{F}^{'}(u_n), u_n \right\rangle \right| \leq \parallel u_n \parallel \ \ \text{for} \ \ n \geq n_0, \ \ \text{or}$$

$$\left| \left\langle \Phi^{'}(u_n), u_n \right\rangle - \left\langle \Psi^{'}(u_n), u_n \right\rangle \right| \leq \parallel u_n \parallel \ \ \text{for} \ \ n \geq n_0,$$

that is

$$\left| \parallel u_n \parallel^p - \int_{\Omega} f(x,u_n)u_n dx \right| \leq \parallel u_n \parallel \ \ \text{for} \ \ n \geq n_0,$$

which gives

$$-\frac{1}{\theta} \parallel u_n \parallel^p + \frac{1}{\theta}\int_\Omega f(x,u_n)u_n dx \leq \frac{1}{\theta} \parallel u_n \parallel \quad \text{for} \quad n \geq n_0. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\left(\frac{1}{p} - \frac{1}{\theta}\right) \parallel u_n \parallel^p - \frac{1}{\theta} \parallel u_n \parallel \leq k,$$

and since $\theta > p > 1$ it results that (u_n) is bounded in X. □

Theorem 3.3. *If either (i) there are numbers* $\theta > p$ *and* $s_1 > 0$ *such that*

$$0 < \theta F(x,s) \leq sf(x,s) \quad \textit{for } x \in \Omega, \ s \geq s_1, \tag{3.9}$$

or (ii) there are numbers $\theta > p$ *and* $s_1 < 0$ *such that*

$$0 < \theta F(x,s) \leq sf(x,s) \quad \textit{for } x \in \Omega, \ s \leq s_1, \tag{3.10}$$

then $\mathcal{F}$ *is unbounded from below.*

Proof. We are going to prove the sufficiency of the condition (i) (the proof for (ii) is similar). Let $u \in X$, $u > 0$ (in fact $i(u) > 0$) be such that *meas* $(M_1(u)) > 0$, where

$$M_1(u) = \{x \in \Omega : \ u(x) \geq s_1\} \quad \text{(in fact } i(u)(x) \geq s_1).$$

We shall show that $\mathcal{F}(\lambda u) \to -\infty$ as $\lambda \to \infty$. For $\lambda \geq 1$ we denote $M_\lambda(u) = \{x \in \Omega : \lambda u(x) \geq s_1\}$. Since $\lambda \geq 1$, it results that $M_1(u) \subset M_\lambda(u)$ and hence *meas*$(M_\lambda(u)) > 0$. By (3.9), for $x \in \Omega$ and $\tau \geq s_1$ we have

$$\frac{\theta}{\tau} \leq \frac{f(x,\tau)}{F(x,\tau)} = \frac{F'_\tau(x,\tau)}{F(x,\tau)},$$

and integrating from s_1 to s we get

$$\ln\left(\frac{s}{s_1}\right)^\theta \leq \ln F(x,s) - \ln F(x,s_1),$$

which implies

$$F(x,s) \geq \gamma(x)s^\theta \quad \text{for} \quad x \in \Omega, \ s \geq s_1, \tag{3.11}$$

where $\gamma(x) = \dfrac{F(x,s_1)}{s_1^\theta} > 0$ and obviously $\gamma \in L^1(\Omega)$. For $\lambda \geq 1$ we have

$$\mathcal{F}(\lambda u) = \frac{\lambda^p}{p} \parallel u \parallel^p - \left(\int_{M_\lambda(u)} F(x,\lambda u)dx + \int_{\Omega \setminus M_\lambda(u)} F(x,\lambda u)dx\right). \tag{3.12}$$

If $x \in M_\lambda(u)$ then $\lambda u(x) \geq s_1$ and using (3.11)

$$F(x,\lambda u(x)) \geq \gamma(x)\lambda^\theta u^\theta.$$

Therefore,

$$\int_{M_\lambda(u)} F(x,\lambda u(x))dx \geq \lambda^\theta \int_{M_\lambda(u)} \gamma(x)u^\theta dx \geq \lambda^\theta \int_{M_1(u)} \gamma(x)u^\theta dx = \lambda^\theta k_1(u), \tag{3.13}$$

with $k_1(u) > 0$. If $x \in \Omega \setminus M_\lambda(u)$ then $\lambda u(x) < s_1$ and using (3.1), we have

$$| F(x,\lambda u(x)) | \leq c_1\lambda^q u^q + c(x) \leq c_1 s_1^q + c(x).$$

Therefore,

$$\left| \int_{\Omega \setminus M_\lambda(u)} F(x, \lambda u(x))dx \right| \leq c_1 s_1^q meas(\Omega) + \int_\Omega c(x)dx = k_2. \tag{3.14}$$

From (3.12),(3.13) and (3.14) we obtain

$$\mathcal{F}(\lambda u) \leq \frac{\lambda^p}{p} \parallel u \parallel^p - \lambda^\theta k_1(u) + k_2 \to -\infty \ \ as \ \ \lambda \to \infty,$$

and the proof is ready. $\square$

Remark 3.1. *By the theorem 3.3, since* $\mathcal{F}$*is unbounded from below it results that for each* $\rho > 0$ *there exists* $e \in X$ *with* $\parallel e \parallel > \rho$ *such that* $\mathcal{F}(e) \leq 0$.

Theorem 3.4. *Assume that the Caratheodory function* $f : \Omega \times R \to R$ *satisfies (i)* $\mid f(x, s) \mid \leq c\left(\mid s \mid^{q-1} + 1\right)$ *for* $x \in \Omega$, $s \in R$, *where* $q \in (1, p^*)$ *and* $c \geq 0$ *constant;*
(ii) $\limsup_{s \to 0} \dfrac{f(x, s)}{\mid s \mid^{p-2} s} < \lambda_1$ *uniformly with* $x \in \Omega$, *where*
$\lambda_1 = \inf \left\{ \dfrac{\parallel v \parallel_X^p}{\parallel v \parallel_{0,p}^p} : v \in X, v \neq 0 \right\}$ *is the smallest eigenvalue for* J_p. *Then there are constants* $\rho, \alpha > 0$ *such that* $\mathcal{F}_{\big| \parallel u \parallel = \rho} \geq \alpha$.

Proof. Let $h : \Omega \to R$ be defined by

$$h(x) = \limsup_{s \to 0} \frac{f(x, s)}{\mid s \mid^{p-2} s}.$$

From (ii) there is $\mu \in (0, \lambda_1)$ such that $h(x) < \mu$ uniformly with $x \in \Omega$. It results that there is some $\delta_\mu > 0$ such that

$$\frac{f(x, s)}{\mid s \mid^{p-2} s} \leq \mu \ \text{ for } x \in \Omega, \ 0 < \mid s \mid < \delta_\mu,$$

or

$$f(x, s) \leq \mu s^{p-1} \ \text{ for } x \in \Omega, \ s \in (0, \delta_\mu), \tag{3.15}$$

$$-\mu \mid s \mid^{p-1} \leq f(x, s) \ \text{ for } x \in \Omega, \ s \in (-\delta_\mu, 0). \tag{3.16}$$

Let us notice that f satisfies $f(x, 0) = 0$, for $x \in \Omega$. Then, from (3.15),(3.16) and by the definition of F we obtain

$$F(x, s) \leq \frac{\mu}{p} \mid s \mid^p \ \text{ for } x \in \Omega, \ 0 < \mid s \mid < \delta_\mu. \tag{3.17}$$

Using (i), we easily see that F satisfies

$$\mid F(x, s) \mid \leq c_1(\mid s \mid^q + 1) \ \text{ for } x \in \Omega, \ s \in R, \tag{3.18}$$

with $c_1 \geq 0$ constant. Let $q_1 \in (\max\{p, q\}, p^*)$. Then from (3.18) there is a constant $c_2 \geq 0$ such that

$$\mid F(x, s) \mid \leq c_2 \mid s \mid^{q_1} \ \text{ for } x \in \Omega, \ \mid s \mid \geq \delta_\mu. \tag{3.19}$$

From (3.17) and (3.19) it follows

$$\mid F(x, s) \mid \leq \frac{\mu}{p} \mid s \mid^p + c_2 \mid s \mid^{q_1} \ \text{ for } x \in \Omega, \ s \in R. \tag{3.20}$$

Using (3.20), the variational characterization of λ_1 and the imbeding $X \to L^{q_1}(\Omega)$ we obtain

$$\begin{aligned}\mathcal{F}(u) &= \frac{1}{p}\parallel u\parallel^p - \int_\Omega F(x,u)dx \geq \frac{1}{p}\parallel u\parallel^p - \frac{\mu}{p}\int_\Omega |\, u\,|^p\, dx - c_2\int_\Omega |\, u\,|^{q_1}\, dx \geq \\ &\geq \frac{1}{p}\parallel u\parallel^p - \frac{\mu}{p}\parallel u\parallel_{0,p}^p - c_3\parallel u\parallel^{q_1} = \\ &= \parallel u\parallel^p\left[\frac{1}{p}\left(1-\mu\frac{\parallel u\parallel_{0,p}^p}{\parallel u\parallel^p}\right) - c_3\parallel u\parallel^{q_1-p}\right] \geq \\ &\geq \parallel u\parallel^p\left[\frac{1}{p}\left(1-\frac{\mu}{\lambda_1}\right) - c_3\parallel u\parallel^{q_1-p}\right] \geq \alpha > 0,\end{aligned}$$

provided $\parallel u\parallel = \rho > 0$ is sufficiently small. □

Now we can prove the most important result of this section.

Theorem 3.5. *Suppose that the hypotheses* $(H_1),(H_2),(H_3)$ *and* (H_4) *hold. Moreover, we assume that the Caratheodory function f satisfies: (i) there is* $q \in (1,p^*)$ *such that*

$$|\, f(x,s)\,| \leq c(|\, s\,|^{q-1}+1) \text{ for } x\in\Omega, s\in R,$$

with $c \geq 0$ *constant; (ii)* $\limsup_{s\to 0}\frac{f(x,s)}{|\, s\,|^{p-2}s} < \lambda_1$ *uniformly with* $x \in \Omega$, *where* λ_1 *is the smallest eigenvalue for* J_p*; (iii) there exist the constants* $\theta > p$ *and* $s_0 > 0$ *such that*

$$0 < \theta F(x,s) \leq sf(x,s) \text{ for } x\in\Omega, |\, s\,| \geq s_0.$$

Then the problem (1.1) has at least a nontrivial solution $u \in X$.

Proof. It is enough to prove that $\mathcal{F}$ has at least a nontrivial critical point $u \in X$. For this we are going to use theorem 2.3. Obviously, $\mathcal{F}(0) = 0$. From $(i),(iii)$ and the theorem 3.2, $\mathcal{F}$ satisfies the (PS) condition. Furthermore, from $(i),(ii)$ and the theorem 3.4 there are constants $\rho,\alpha > 0$ such that $\mathcal{F}\big|_{\parallel u\parallel=\rho} \geq \alpha$. Finally, from $(i),(iii)$ and the theorem 3.3 (also see the remark 3.1) there is an element $e \in X$, $\parallel e\parallel > \rho$ such that $\mathcal{F}(e) \leq 0$ and the proof is complete. □

3.3 Existence of the solution using a direct variational method

Theorem 3.6. *Assume that the hypotheses* $(H_1),(H_2),(H_3)$ *and* (H_4) *hold. Moreover, assume that the Caratheodory function f satisfies the growth condition (1.2) with* $1 < q < p$. *Then the problem (1.1) has at least a solution* $u \in X$.

Proof. For $u \in X$, using (3.1) we have

$$\mathcal{F}(u) = \frac{1}{p}\parallel u\parallel^p - \int_\Omega F(x,u)dx \geq \frac{1}{p}\parallel u\parallel^p - c_1\parallel u\parallel_{o,q}^q - \int_\Omega c(x)dx.$$

By the imbedding $X \to L^q(\Omega)$ it results

$$\mathcal{F}(u) \geq \frac{1}{p}\parallel u\parallel^p - c_2\parallel u\parallel^q - c_3,$$

where $c_2, c_3 > 0$ are constants. Since $p > q > 1$ it follows that $\mathcal{F}$ is coercive and bounded from below on X. Let $l = \inf_X \mathcal{F}$ and $(u_n) \subset X$ such that $\mathcal{F}(u_n) \to l$. Since $\mathcal{F}$ is coercive it results that (u_n) is bounded in X and by the reflexivity of X, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ (weakly) in X. Since the imbedding $X \to L^q(\Omega)$ is compact, it results that $u_n \to u$ (strongly) in $L^q(\Omega)$. We have

$$\mathcal{F}(u) = \Phi(u) - \Psi(u) = Fc(u) = \frac{1}{p} \parallel u \parallel^p - \int_\Omega F(x,u)dx.$$

Since Φ is C^1 on X and it is convexe, it results that Φ is weakly lower semicontinuous and then

$$\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u).$$

From the continuity of Ψ on $L^q(\Omega)$ we have

$$\Psi(u_n) \to \Psi(u),$$

and then

$$l = \liminf_{n\to\infty} \mathcal{F}(u_n) \geq \liminf_{n\to\infty} \Phi(u_n) - \Psi(u) \geq \Phi(u) - \Psi(u) = \mathcal{F}(u) \geq l,$$

which implies $\mathcal{F}(u) = l$. Conseqently, u is a minimum point for $\mathcal{F}$ and then it is a critical point, which means that it is a solution for the problem (1.1). □

Remark 3.2. *Another proof of the theorem (3.6) can be given by using the Fredholm - type result recalled in theorem 2.4.*

Indeed, we apply the theorem 2.4, choosing $Y = X^*, T = J_p, S = N_f$ (by which means $i' N_f i : X \to X^*$). Since the imbedding $X \to L^q(\Omega)$, is compact it results that N_f is compact. By the proposition 2.5, T is bijective and T^{-1} continuous. For $u \in X$ we have

$$\begin{aligned} \parallel \lambda J_p u - N_f u \parallel_* &\geq \parallel \lambda J_p u \parallel_* - \parallel N_f u \parallel_* \geq |\lambda| \parallel u \parallel^{p-1} - c \parallel u \parallel_{0,q}^{q-1} - c_1 \geq \\ &\geq |\lambda| \parallel u \parallel^{p-1} - c_2 \parallel u \parallel^{q-1} - c_1 \to \infty \text{ as } \parallel u \parallel \to \infty, \end{aligned}$$

because $p > q > 1$. To verify the hypothesis (iv) we use the remark 2.2 following the calculus used in 3.1.

Remark 3.3. *The advantage of using the Mountain Pass Theorem in order to prove the existence of the problem's (1.1) solution is given by the fact that we show its nontriviality.*

Let us remark yet that in the particular case of $f(x,0) \neq 0$, it results that $u = 0$ is not a solution , hence the nontriavility of the problem's (1.1) solution is provided by the theorems 3.1 and 3.6, too.

3.4 Multiple solutions

We use the following result:

Proposition 3.2. *Assume that the Caratheodory function $f : \Omega \times R \to R$ satisfies (i) there is $q \in (1, p^*)$ such that*

$$|f(x,s)| \leq c(|s|^{q-1} + 1) \;\; \text{for } x \in \Omega, s \in R,$$

with $c \geq 0$ constant. (ii) there are some numbers $\theta > p$ and $s_0 > 0$ such that

$$0 < \theta F(x,s) \leq sf(x,s) \;\; for\; x \in \Omega, |s| \geq s_0.$$

Then, if X_1 is a finite dimensional subspace of X, the set $S = \{v \in X_1 : \mathcal{F}(v) \geq 0\}$ is bounded in X.

Proof. From (i) F satisfies

$$|\, F(x,s)\,| \leq c_1(|\, s\,|^q + 1) \text{ for } x \in \Omega,\ s \in R, \tag{3.21}$$

with $c_1 \geq 0$ constant. We show that there is $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ in Ω such that

$$F(x,s) \geq \gamma(x)\,|\, s\,|^\theta \text{ for } x \in \Omega,\ |\, s\,| \geq s_0. \tag{3.22}$$

So, as in the proof of the theorem 3.3 we get

$$F(x,s) \geq \gamma_1(x) s^\theta \text{ for } x \in \Omega,\ s \geq s_0, \tag{3.23}$$

where $\gamma_1(x) = \dfrac{F(x,s_0)}{s_0^\theta}$. By (3.21) it results that $\gamma_1 \in L^\infty(\Omega)$ and (ii) yields $\gamma_1 > 0$ on Ω. Analogously, we have

$$F(x,s) \geq \gamma_2(x)\,|\, s\,|^\theta \text{ for } x \in \Omega,\ s \leq -s_0, \tag{3.24}$$

where $\gamma_2(x) = \dfrac{F(x,-s_0)}{s_0^\theta}$. Again $\gamma_2 \in L^\infty(\Omega)$ and $\gamma_2 > 0$ on Ω. Therefore, (3.22) holds with $\gamma(x) = \min\{\gamma_1(x), \gamma_2(x)\}$ for $x \in \Omega$, as assumed. We are going to show that $\mathcal{F}$ satisfies

$$\mathcal{F}(v) \leq \frac{1}{p}\,\|\, v\,\|^p - \int_\Omega \gamma(x)\,|\, v\,|^\theta\, dx + K, \text{ for all } v \in X, \tag{3.25}$$

with K constant. For $v \in X$ we denote $\Omega_v = \{x \in \Omega : |\, v(x)\,| < s_0\}$. By (3.21) we have

$$\int_{\Omega_v} F(x,v)dx \geq -c_1 \int_{\Omega_v} (|\, v\,|^p + 1)dx \geq -c_1 \int_\Omega (s_0^q + 1)dx = -c_1(s_0^q + 1)meas(\Omega) = k_1,$$

and by (3.22) it holds

$$\int_{\Omega\setminus\Omega_v} F(x,v)dx \geq \int_{\Omega\setminus\Omega_v} \gamma(x)\,|\, v\,|^\theta\, dx.$$

Then

$$\begin{aligned}
\mathcal{F}(v) &= \frac{1}{p}\,\|\, v\,\|^p - \left(\int_{\Omega_v} F(x,v)dx + \int_{\Omega\setminus\Omega_v} F(x,v)dx\right) \leq \\
&\leq \frac{1}{p}\,\|\, v\,\|^p - \int_{\Omega\setminus\Omega_v} \gamma(x)\,|\, v\,|^\theta\, dx - k_1 = \\
&= \frac{1}{p}\,\|\, v\,\|^p - \int_\Omega \gamma(x)\,|\, v\,|^\theta\, dx + \int_{\Omega_v} \gamma(x)\,|\, v\,|^\theta - k_1 \leq \\
&\leq \frac{1}{p}\,\|\, v\,\|^p - \int_\Omega \gamma(x)\,|\, v\,|^\theta + K,
\end{aligned}$$

where $K = \|\,\gamma\,\|_{0,\infty} s_0^q meas(\Omega) - k_1$, and (3.25) is proved. The functional $\|\cdot\|_\gamma : X \to R$ defined by

$$\|\, v\,\|_\gamma = \left(\int_\Omega \gamma(x)\,|\, v\,|^\theta\, dx\right)^{\frac{1}{\theta}},$$

is a norm on X. On the finite dimensional subspace X_1 the norms $\|\cdot\|_X$ and $\|\cdot\|_\gamma$ being equivalent, there is a constant $\widetilde{K} = \widetilde{K}\,(X_1) > 0$ such that

$$\|\, v\,\|_X \leq \widetilde{K} \left(\int_\Omega \gamma(x)\,|\, v\,|^\theta\, dx\right)^{\frac{1}{\theta}} \text{ for all } v \in X_1.$$

Consequently, by (3.25) on X_1 it holds

$$\begin{aligned} \mathcal{F}(v) &\leq \frac{1}{p}\tilde{K}^{p}\left(\int_\Omega \gamma(x) \mid v \mid^\theta dx\right)^{\frac{p}{\theta}} - \int_\Omega \gamma(x) \mid v \mid^\theta dx + K = \\ &= \frac{1}{p}\tilde{K}^{p} \parallel v \parallel_\gamma^p - \parallel v \parallel_\gamma^\theta + K. \end{aligned}$$

Therefore

$$\frac{1}{p}\tilde{K}^{p} \parallel v \parallel_\gamma^p - \parallel v \parallel_\gamma^\theta + K \geq 0 \text{ for all } v \in S,$$

and since $\theta > p$ it results that S is bounded. □

Now we can state

Theorem 3.7. *Suppose that the hypotheses $(H_1), (H_2), (H_3)$ and (H_4) are satisfied. Moreover, assume that the Caratheodory function f is odd in the second argument :*
$f(x, -s) = -f(x, s)$ and the conditions $(i), (ii), (iii)$ of the theorem (3.5) are satisfied. Then the problem (1.1) has an unbounded sequence of solutions.

Proof. Since f is odd it results that $\mathcal{F}$ is even. Obviously, $\mathcal{F}(0) = 0$. From $(i), (iii)$ and the theorem 3.2, $\mathcal{F}$ satisfies the (PS) condition. Furthemore, from $(i), (ii)$ and the theorem 3.4, there are constants $\rho, \alpha > 0$ such that $\mathcal{F}\big|_{\parallel u \parallel = \rho} \geq \alpha$. The proposition 3.2 and $(i), (iii)$ show that the set $\{v \in X_1 : \mathcal{F}(v) \geq 0\}$ is bounded in X, whenever X_1 is a finite dimensional subspace of X. By the theorem 2.5 it results that $\mathcal{F}$ possesses an unbounded sequence of critical values, so that the problem (1.1) has an unbounded sequence of solutions. □

4 Examples

Example 4.1. If $X = W_0^{1,p}(\Omega)$ then $J_p = -\Delta_p$ and the solutions set of the equation $J_p u = N_f u$ is the same with the Dirichlet problem's (1.3) solutions set (see the Introduction). Consequently, the existence results given in the section 3 imply the existence results obtained in [1] for the Dirichlet problem (1.3).

Example 4.2. We consider $X = W^{1,p}(\Omega)$, endowed with the norm

$$\||| u |||_{1,p}^p = \parallel u \parallel_{0,p}^p + \parallel \mid \nabla u \mid \parallel_{0,p}^p \text{ for all } u \in W^{1,p}(\Omega),$$

which is equivalent with the standard norm on the space $W^{1,p}(\Omega)$ (see[14, 4]). In this case, the duality mapping J_p on $\left(W^{1,p}(\Omega), ||| \cdot |||_{1,p}\right)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$ is defined by

$$\begin{aligned} &J_p : \left(W^{1,p}(\Omega), ||| \cdot |||_{1,p}\right) \to \left(W^{1,p}(\Omega), ||| \cdot |||_{1,p}\right)^*, \\ &J_p u = -\Delta_p u + \mid u \mid^{p-2} u \quad \text{for all} \quad u \in W^{1,p}(\Omega) \end{aligned} \tag{4.1}$$

(see [4]). By weak solution of the Neumann problem (1.7), (1.8) we mean an element $u \in W^{1,p}(\Omega)$ which satisfies

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \nabla v dx + \int_\Omega \mid u \mid^{p-2} uv dx = \int_\Omega f(x, u) v dx \quad \text{for all} \quad v \in W^{1,p}(\Omega). \tag{4.2}$$

Obviously, $u \in W^{1,p}(\Omega)$ is a solution for the problem (1.7), (1.8) in the meaning of (4.2) if and only if

$$J_p u = \left(i^{'} N_f i\right) u,$$

where J_p is given by (4.1), $i : W^{1,p}(\Omega) \to L^q(\Omega)$ is the compact imbedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ and $i' : L^{q'}(\Omega) \to \left(W^{1,p}(\Omega), \|\!| \cdot \|\!|_{1,p}\right)^*$ is its adjoinct. So, we are in the functional framework (H_2), (H_3) about X. Indeed, $\left(W^{1,p}(\Omega), \|\!| \cdot \|\!|_{1,p}\right)$ is a smooth reflexive Banach space, compactly imbedded in the space $L^q(\Omega)$, $J_p : \left(W^{1,p}(\Omega), \|\!| \cdot \|\!|_{1,p}\right) \to \left(W^{1,p}(\Omega), \|\!| \cdot \|\!|_{1,p}\right)^*$ is single valued, continuous and satisfies the (S_+) condition (see [4]). Therefore, the existence results obtained in the section 3 imply the existence results obtained in [4] for the Neumann problem (1.7), (1.8).

5 Conclusion

Taking into account the proofs written above, the existence results have been obtained for the operational equation (1.1), using variational and topological methods. In particular, these results allow a unitary approach to the Dirichlet (1.3) and Neumann (1.7, 1.8) problems, which are operationally written in the same form, using the equation (1.1).

Competing Interests

Author has declared that no competing interests exist.

References

[1] Dinca G, Jebelean P, Mahwin J. Variational and topological methods for dirichlet problems with p-Laplacian. Portugaliae Mathematica. 2001;58(3):339-378.

[2] Lions JL. Quelques méthodes de résolution des problèmes aux limites non lineaires. Dunod-Gauthier, Villars, Paris; 1969.

[3] De Figueiredo DG. Lectures on Ekeland variational principle with applications and detours. Tata Institut of Fundamental Research, Springer-Verlag; 1989.

[4] Cringanu J. Variational and topological methods for Neumann problems with p-Laplacian. Communications on Applied Nonlinear Analysis. 2004;11:1-38.

[5] Browder FE. Problèmes non-lineaires. Les Presses de l'Université de Montreal; 1964.

[6] Cioranescu I. Duality mapping in nonlinear functional analysis. Publishing House of Romanian Academy, Bucharest; 1974.

[7] Dinca G, Jebelean P. Some existence results for a class of nonlinear equations involving a duality mapping. Nonlinear Analysis. 2001;46:347-363.

[8] Dinca G. A Fredholm-type result for a couple of nonlinear operator. C. R. Acad. Sci. Paris, t., Serie I. 2001;333:415-419.

[9] Ambrosetti A, Rabinowitz PH. Dual variational methods in critical points theory and applications. J. Funct. Anal. 1973;14:349-381.

[10] Cringanu J, Dinca G. Multiple solutions for a class of nonlinear equations involving a duality mapping. Differential and Integral Equations. 2008;21(3-4):265-284.

[11] Kavian O. Introduction a la thérie des points critique. Springer Verlag; 1993.

[12] Mawhin J, Willem M. Critical point theory and Hamiltonian systems. Springer; 1989.

[13] Rabinowitz PH. Minimax methods in critical point theory with applications to differential equations. CBMS, Reg. Ser. Math. 65, A.M.S. - Providence, R.I.; 1986.

[14] ADAMS RA. Sobolev spaces. Academic Press; 1975.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/62190