



Binomial Transform of the Generalized Tribonacci Sequence

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Abstract

In this paper, we define the binomial transform of the generalized Tribonacci sequence and as special cases, the binomial transform of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences will be introduced. We investigate their properties in details. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these binomial transforms. Moreover, we give some identities and matrices related with these binomial transforms.

Keywords: Binomial transform; Tribonacci sequence; Tribonacci numbers; Tribonacci-Lucas sequence; Tribonacci-Lucas numbers; binomial transform of Tribonacci sequence; 3-step Fibonacci sequence.

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1 Introduction and Preliminaries

In this paper, we introduce the binomial transform of the generalized Tribonacci sequence and we investigate, in detail, six special cases which we call them the binomial transform of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized 3-step Fibonacci sequence.

The generalized 3-step Fibonacci sequence (also called the generalized Tribonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$$

(or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \quad (1.1)$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1,2,3,4,5,6,7,8,9,10,11,12,13].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n .

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \quad (1.2)$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r, s, t) = \frac{r}{3} + A + B, \\ \beta &= \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta} \right)^{1/3}, \quad B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta} \right)^{1/3}, \\ \Delta &= \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

If $\Delta(r, s, t) > 0$, then (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that the generalized 3-step Fibonacci numbers (the generalized Tribonacci numbers) can be expressed, for all integers n , using Binet's formula

$$W_n = \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \quad (1.3)$$

where

$$p_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \quad p_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \quad p_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n , for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n .

Lemma 1.1. Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized 3-step Fibonacci sequence (the generalized Tribonacci sequence) $\{W_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \quad (1.4)$$

We next find Binet's formula of the generalized 3-step Fibonacci sequence (the generalized Tribonacci sequence) $\{W_n\}$ by the use of generating function for W_n .

Theorem 1.2. (Binet's formula of the generalized 3-step Fibonacci numbers (the generalized Tribonacci numbers)) For all integers n , we have

$$W_n = \frac{q_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{q_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{q_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \quad (1.5)$$

where

$$\begin{aligned} q_1 &= W_0 \alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ q_2 &= W_0 \beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ q_3 &= W_0 \gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Note that from (1.3) and (1.5) we have

$$\begin{aligned} W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0 &= W_0 \alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0 &= W_0 \beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0 &= W_0 \gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

In this paper, we consider the case $r = 1, s = 1, t = 1$ and in this case we write $V_n = W_n$. So, the generalized Tribonacci sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2)\}_{n \geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} \quad (1.6)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.6) holds for all integer n .

(1.3) can be used to obtain Binet's formula of generalized Tribonacci numbers. Binet's formula of generalized Tribonacci numbers can be given as

$$V_n = \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \quad (1.7)$$

where

$$p_1 = V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0, \quad (1.8)$$

$$p_2 = V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0, \quad (1.9)$$

$$p_3 = V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0. \quad (1.10)$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x^2 - x - 1 = 0.$$

Moreover,

$$\begin{aligned} \alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \beta &= \frac{1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \gamma &= \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}}{3}, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Now, we define four new special cases of the sequence $\{V_n\}$ besides the well known Tribonacci sequence $\{T_n\}_{n \geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n \geq 0}$.

Tribonacci sequence $\{T_n\}_{n \geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n \geq 0}$, Tribonacci-Perrin sequence $\{M_n\}_{n \geq 0}$, modified Tribonacci sequence $\{U_n\}_{n \geq 0}$, modified Tribonacci-Lucas sequence $\{G_n\}_{n \geq 0}$ and adjusted Tribonacci-Lucas sequence $\{H_n\}_{n \geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad T_0 = 0, T_1 = 1, T_2 = 1, \quad (1.11)$$

$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3, \quad (1.12)$$

$$M_{n+3} = M_{n+2} + M_{n+1} + M_n, \quad M_0 = 3, M_1 = 0, M_2 = 2, \quad (1.13)$$

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \quad U_0 = 1, U_1 = 1, U_2 = 1. \quad (1.14)$$

$$G_{n+3} = G_{n+2} + G_{n+1} + G_n, \quad G_0 = 4, G_1 = 4, G_2 = 10, \quad (1.15)$$

$$H_{n+3} = H_{n+2} + H_{n+1} + H_n, \quad H_0 = 4, H_1 = 2, H_2 = 0, \quad (1.16)$$

The sequences $\{T_n\}_{n \geq 0}$, $\{K_n\}_{n \geq 0}$, $\{M_n\}_{n \geq 0}$, $\{U_n\}_{n \geq 0}$, $\{G_n\}_{n \geq 0}$, and $\{H_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \quad (1.17)$$

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \quad (1.18)$$

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} + M_{-(n-3)}, \quad (1.19)$$

$$U_{-n} = -U_{-(n-1)} - U_{-(n-2)} + U_{-(n-3)}, \quad (1.20)$$

$$G_{-n} = -G_{-(n-1)} - G_{-(n-2)} + G_{-(n-3)}, \quad (1.21)$$

$$H_{-n} = -H_{-(n-1)} - H_{-(n-2)} + H_{-(n-3)}, \quad (1.22)$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.11)-(1.16) hold for all integer n . For more details on the generalized Tribonacci numbers, see Soykan [15].

T_n is the sequence A000073 in [16], K_n is the sequence A001644 in [16] and U_n is the sequence A000213 in [16].

For all integers n , Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers (using initial conditions in (1.8)-(1.10))

can be expressed using Binet's formulas as

$$\begin{aligned} T_n &= \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}, \\ K_n &= \alpha^n + \beta^n + \gamma^n, \\ M_n &= \frac{(2\alpha + 3)\alpha^{n-1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(2\beta + 3)\beta^{n-1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{(2\gamma + 3)\gamma^{n-1}}{(\gamma - \alpha)(\gamma - \beta)}, \\ U_n &= \frac{(\alpha^2 + 1)\alpha^{n-1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(\beta^2 + 1)\beta^{n-1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{(\gamma^2 + 1)\gamma^{n-1}}{(\gamma - \alpha)(\gamma - \beta)}, \\ G_n &= (\alpha + 1)\alpha^n + (\beta + 1)\beta^n + (\gamma + 1)\gamma^n, \\ H_n &= (\alpha - 1)^2\alpha^n + (\beta - 1)^2\beta^n + (\gamma - 1)^2\gamma^n, \end{aligned}$$

respectively, see, Soykan [15] for more details.

2 Binomial Transform of the Generalized Tribonacci Sequence V_n

In [17, p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n) , its binomial transform (\hat{a}_n) may be defined by the rule

$$\hat{a}_n = \sum_{i=0}^n \binom{n}{i} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \sum_{i=0}^n \binom{n}{i} (-1)^{i+1} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^n \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, for example, [18,19,20,21] and references therein.

In this section, we define the binomial transform of the generalized Tribonacci sequence V_n and as special cases the binomial transform of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences will be introduced.

Definition 2.1. The binomial transform of the generalized Tribonacci sequence V_n is defined by

$$b_n = \hat{V}_n = \sum_{i=0}^n \binom{n}{i} V_i.$$

The few terms of b_n are

$$\begin{aligned} b_0 &= \sum_{i=0}^0 \binom{0}{i} V_i = V_0, \\ b_1 &= \sum_{i=0}^1 \binom{1}{i} V_i = V_0 + V_1, \\ b_2 &= \sum_{i=0}^2 \binom{2}{i} V_i = V_0 + 2V_1 + V_2. \end{aligned}$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences are defined as follows: The binomial transform of the Tribonacci sequence T_n is

$$\widehat{T}_n = \sum_{i=0}^n \binom{n}{i} T_i,$$

the binomial transform of the Tribonacci-Lucas sequence K_n is

$$\widehat{K}_n = \sum_{i=0}^n \binom{n}{i} K_i,$$

the binomial transform of the Tribonacci-Perrin sequence M_n is

$$\widehat{M}_n = \sum_{i=0}^n \binom{n}{i} M_i,$$

the binomial transform of the modified Tribonacci sequence U_n is

$$\widehat{U}_n = \sum_{i=0}^n \binom{n}{i} U_i,$$

the binomial transform of the modified Tribonacci-Lucas sequence G_n is

$$\widehat{G}_n = \sum_{i=0}^n \binom{n}{i} G_i,$$

the binomial transform of the adjusted Tribonacci-Lucas sequence H_n is

$$\widehat{H}_n = \sum_{i=0}^n \binom{n}{i} H_i.$$

Lemma 2.1. For $n \geq 0$, the binomial transform of the generalized Tribonacci sequence V_n satisfies the following relation:

$$b_{n+1} = \sum_{i=0}^n \binom{n}{i} (V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned}
 b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\
 &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\
 &= V_0 + \sum_{i=1}^n \binom{n}{i} V_i + \sum_{i=0}^n \binom{n}{i} V_{i+1} \\
 &= \sum_{i=0}^n \binom{n}{i} V_i + \sum_{i=0}^n \binom{n}{i} V_{i+1} \\
 &= \sum_{i=0}^n \binom{n}{i} (V_i + V_{i+1}).
 \end{aligned}$$

This completes the proof. \square

Remark 2.1. From the last Lemma, we see that

$$b_{n+1} = b_n + \sum_{i=0}^n \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized Tribonacci sequence.

Theorem 2.2. For $n \geq 0$, the binomial transform of the generalized Tribonacci sequence V_n satisfies the following recurrence relation:

$$b_{n+3} = 4b_{n+2} - 4b_{n+1} + 2b_n. \tag{2.1}$$

Proof. To show (2.1), writing

$$b_{n+3} = A \times b_{n+2} + B \times b_{n+1} + C \times b_n$$

and taking the values $n = 0, 1, 2$ and then solving the system of equations

$$\begin{aligned}
 b_3 &= A \times b_2 + B \times b_1 + C \times b_0 \\
 b_4 &= A \times b_3 + B \times b_2 + C \times b_1 \\
 b_5 &= A \times b_4 + B \times b_3 + C \times b_2
 \end{aligned}$$

we find that $A = 4, B = -4, C = 2$. \square

The sequence $\{b_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = 2b_{-n+1} - 2b_{-n+2} + \frac{1}{2}b_{-n+3}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (2.1) holds for all integer n .

Note that the recurrence relation (2.1) is independent from initial values. So,

$$\begin{aligned}
 \widehat{T}_{n+3} &= 4\widehat{T}_{n+2} - 4\widehat{T}_{n+1} + 2\widehat{T}_n, \\
 \widehat{K}_{n+3} &= 4\widehat{K}_{n+2} - 4\widehat{K}_{n+1} + 2\widehat{K}_n, \\
 \widehat{M}_{n+3} &= 4\widehat{M}_{n+2} - 4\widehat{M}_{n+1} + 2\widehat{M}_n, \\
 \widehat{U}_{n+3} &= 4\widehat{U}_{n+2} - 4\widehat{U}_{n+1} + 2\widehat{U}_n, \\
 \widehat{G}_{n+3} &= 4\widehat{G}_{n+2} - 4\widehat{G}_{n+1} + 2\widehat{G}_n, \\
 \widehat{H}_{n+3} &= 4\widehat{H}_{n+2} - 4\widehat{H}_{n+1} + 2\widehat{H}_n.
 \end{aligned}$$

The first few terms of the binomial transform of the generalized Tribonacci sequence with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few binomial transform (terms) of the generalized Tribonacci sequence

n	b_n	b_{-n}
0	V_0	...
1	$V_0 + V_1$	$\frac{1}{2}V_0 - V_1 + \frac{1}{2}V_2$
2	$V_0 + 2V_1 + V_2$	$V_2 - \frac{3}{2}V_1 - \frac{1}{2}V_0$
3	$2V_0 + 4V_1 + 4V_2$	$V_2 - V_1 - \frac{3}{2}V_0$
4	$6V_0 + 10V_1 + 12V_2$	$\frac{1}{2}V_1 - \frac{7}{4}V_0 + \frac{1}{4}V_2$
5	$18V_0 + 28V_1 + 34V_2$	$\frac{9}{4}V_1 - \frac{3}{4}V_0 - V_2$
6	$52V_0 + 80V_1 + 96V_2$	$\frac{5}{4}V_0 + 3V_1 - 2V_2$
7	$148V_0 + 228V_1 + 272V_2$	$\frac{25}{8}V_0 + \frac{7}{4}V_1 - \frac{15}{8}V_2$
8	$420V_0 + 648V_1 + 772V_2$	$\frac{27}{8}V_0 - \frac{11}{8}V_1 - \frac{1}{4}V_2$
9	$1192V_0 + 1840V_1 + 2192V_2$	$\frac{9}{8}V_0 - \frac{19}{4}V_1 + \frac{9}{4}V_2$
10	$3384V_0 + 5224V_1 + 6224V_2$	$\frac{65}{16}V_2 - \frac{47}{8}V_1 - \frac{47}{16}V_0$
11	$9608V_0 + 14832V_1 + 17672V_2$	$\frac{7}{2}V_2 - \frac{47}{16}V_1 - \frac{103}{16}V_0$
12	$27280V_0 + 42112V_1 + 50176V_2$	$\frac{7}{2}V_1 - \frac{103}{16}V_0$
13	$77456V_0 + 119568V_1 + 142464V_2$	$\frac{159}{16}V_1 - \frac{47}{32}V_0 - \frac{159}{32}V_2$

The first few terms of the binomial transform numbers of the Tribonacci , Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas sequences with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few binomial transform (terms)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
\hat{T}_n	0	1	3	8	22	62	176	500	1420	4032	11448	32504	92288	262032
\hat{T}_{-n}		$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{5}{4}$	1	$-\frac{1}{8}$	$-\frac{13}{8}$	$-\frac{5}{2}$	$-\frac{29}{16}$	$\frac{9}{16}$	$\frac{7}{2}$	$\frac{159}{32}$
\hat{K}_n	3	4	8	22	64	184	524	1488	4224	11992	34048	96672	274480	779328
\hat{K}_{-n}		2	0	$-\frac{5}{2}$	-4	-3	$\frac{3}{4}$	$\frac{11}{2}$	8	$\frac{43}{8}$	$-\frac{5}{2}$	$-\frac{47}{4}$	$-\frac{253}{16}$	$-\frac{75}{8}$
\hat{M}_n	3	3	5	14	42	122	348	988	2804	7960	22600	64168	182192	517296
\hat{M}_{-n}		$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{19}{4}$	$-\frac{17}{4}$	$-\frac{1}{4}$	$\frac{45}{8}$	$\frac{77}{8}$	$\frac{63}{8}$	$-\frac{11}{16}$	$-\frac{197}{16}$	$-\frac{309}{16}$	$-\frac{459}{32}$
\hat{U}_n	1	2	4	10	28	80	228	648	1840	5224	14832	42112	119568	339488
\hat{U}_{-n}		0	-1	$-\frac{3}{2}$	-1	$\frac{1}{2}$	$\frac{9}{4}$	3	$\frac{7}{4}$	$-\frac{11}{8}$	$-\frac{19}{4}$	$-\frac{47}{8}$	$-\frac{47}{2}$	$\frac{7}{2}$
\hat{G}_n	4	8	22	64	184	524	1488	4224	11992	34048	96672	274480	779328	2212736
\hat{G}_{-n}		3	2	0	$-\frac{5}{2}$	-4	-3	$\frac{3}{4}$	$\frac{11}{2}$	8	$\frac{43}{8}$	$-\frac{5}{2}$	$-\frac{47}{4}$	$-\frac{253}{16}$
\hat{H}_n	4	6	8	16	44	128	368	1048	2976	8448	23984	68096	193344	548960
\hat{H}_{-n}		0	-5	-8	-6	$\frac{3}{2}$	11	16	$\frac{43}{4}$	-5	$-\frac{47}{2}$	$-\frac{253}{8}$	$-\frac{75}{4}$	14

(1.3) can be used to obtain Binet’s formula of the binomial transform of generalized Tribonacci numbers. Binet’s formula of the binomial transform of generalized Tribonacci numbers can be given as

$$b_n = \frac{c_1\theta_1^n}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{c_2\theta_2^n}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{c_3\theta_3^n}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)} \tag{2.2}$$

where

$$\begin{aligned} c_1 &= b_2 - (\theta_2 + \theta_3)b_1 + \theta_2\theta_3b_0 = (V_0 + 2V_1 + V_2) - (\theta_2 + \theta_3)(V_0 + V_1) + \theta_2\theta_3V_0, \\ c_2 &= b_2 - (\theta_1 + \theta_3)b_1 + \theta_1\theta_3b_0 = (V_0 + 2V_1 + V_2) - (\theta_1 + \theta_3)(V_0 + V_1) + \theta_1\theta_3V_0, \\ c_3 &= b_2 - (\theta_1 + \theta_2)b_1 + \theta_1\theta_2b_0 = (V_0 + 2V_1 + V_2) - (\theta_1 + \theta_2)(V_0 + V_1) + \theta_1\theta_2V_0. \end{aligned}$$

Here, θ_1 , θ_2 and θ_3 are the roots of the cubic equation $x^3 - 4x^2 + 4x - 2 = 0$. Moreover,

$$\begin{aligned}\theta_1 &= \frac{4}{3} + \left(\frac{19}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} + \left(\frac{19}{27} - \sqrt{\frac{11}{27}}\right)^{1/3}, \\ \theta_2 &= \frac{4}{3} + \omega \left(\frac{19}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} + \omega^2 \left(\frac{19}{27} - \sqrt{\frac{11}{27}}\right)^{1/3}, \\ \theta_3 &= \frac{4}{3} + \omega^2 \left(\frac{19}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} + \omega \left(\frac{19}{27} - \sqrt{\frac{11}{27}}\right)^{1/3},\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}\theta_1 + \theta_2 + \theta_3 &= 4, \\ \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 &= 4, \\ \theta_1\theta_2\theta_3 &= 2.\end{aligned}$$

For all integers n , (Binet's formulas of) binomial transforms of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$\begin{aligned}\widehat{T}_n &= \frac{(-1 + \theta_1)\theta_1^n}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{(-1 + \theta_2)\theta_2^n}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{(-1 + \theta_3)\theta_3^n}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{K}_n &= \theta_1^n + \theta_2^n + \theta_3^n, \\ \widehat{M}_n &= \frac{(3\theta_1^2 - 7\theta_1 + 6)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{(3\theta_2^2 - 7\theta_2 + 6)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{(3\theta_3^2 - 7\theta_3 + 6)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{U}_n &= \frac{2(\theta_1 - 1)^2\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(\theta_2 - 1)^2\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(\theta_3 - 1)^2\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{G}_n &= \frac{2(4\theta_1^2 - 5\theta_1 + 4)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(4\theta_2^2 - 5\theta_2 + 4)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(4\theta_3^2 - 5\theta_3 + 4)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{H}_n &= \frac{2(3\theta_1 - 2)(\theta_1 - 2)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(3\theta_2 - 2)(\theta_2 - 2)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(3\theta_3 - 2)(\theta_3 - 2)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)},\end{aligned}$$

respectively.

3 Generating Functions and Obtaining Binet Formula of Binomial Transform from Generating Function

The generating function of the binomial transform of the generalized Tribonacci sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized Tribonacci sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n .

Lemma 3.1. Suppose that $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized Tribonacci sequence $\{V_n\}_{n \geq 0}$. Then, $f_{b_n}(x)$ is given by

$$f_{b_n}(x) = \frac{V_0 + (V_1 - 3V_0)x + (V_0 - 2V_1 + V_2)x^2}{1 - 4x + 4x^2 - 2x^3}. \tag{3.1}$$

Proof. Using Lemma 1.1, we obtain

$$\begin{aligned} f_{b_n}(x) &= \frac{b_0 + (b_1 - 4b_0)x + (b_2 - 4b_1 + 4b_0)x^2}{1 - 4x + 4x^2 - 2x^3} \\ &= \frac{V_0 + (V_1 - 3V_0)x + (V_0 - 2V_1 + V_2)x^2}{1 - 4x + 4x^2 - 2x^3} \end{aligned}$$

where

$$\begin{aligned} b_0 &= V_0, \\ b_1 &= V_0 + V_1, \\ b_2 &= V_0 + 2V_1 + V_2. \quad \square \end{aligned}$$

Note that P. Barry shows in [22] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right)$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum_{i=0}^n \binom{n}{i} a_i$. In our case, since

$$A(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3}, \quad \text{see [15]},$$

we obtain

$$\begin{aligned} S(x) &= \frac{1}{1-x} \frac{V_0 + (V_1 - V_0)\frac{x}{1-x} + (V_2 - V_1 - V_0)\left(\frac{x}{1-x}\right)^2}{1 - \frac{x}{1-x} - \left(\frac{x}{1-x}\right)^2 - \left(\frac{x}{1-x}\right)^3} \\ &= \frac{V_0 + (V_1 - 3V_0)x + (V_0 - 2V_1 + V_2)x^2}{1 - 4x + 4x^2 - 2x^3}. \end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 3.2. *Generating functions of the binomial transform of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{T}_n x^n &= \frac{x - x^2}{1 - 4x + 4x^2 - 2x^3}, \\ \sum_{n=0}^{\infty} \widehat{K}_n x^n &= \frac{3 - 8x + 4x^2}{1 - 4x + 4x^2 - 2x^3}, \\ \sum_{n=0}^{\infty} \widehat{M}_n x^n &= \frac{3 - 9x + 5x^2}{1 - 4x + 4x^2 - 2x^3}, \\ \sum_{n=0}^{\infty} \widehat{U}_n x^n &= \frac{1 - 2x}{1 - 4x + 4x^2 - 2x^3}, \\ \sum_{n=0}^{\infty} \widehat{G}_n x^n &= \frac{4 - 8x + 6x^2}{1 - 4x + 4x^2 - 2x^3}, \\ \sum_{n=0}^{\infty} \widehat{H}_n x^n &= \frac{4 - 10x}{1 - 4x + 4x^2 - 2x^3}, \end{aligned}$$

respectively.

We next find Binet's formula of the Binomial transform of the generalized Tribonacci numbers $\{V_n\}$ by the use of generating function for b_n .

Theorem 3.3. (Binet's formula of the Binomial transform of the generalized Tribonacci numbers)

$$b_n = \frac{d_1\theta_1^n}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{d_2\theta_2^n}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{d_3\theta_3^n}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)} \quad (3.2)$$

where

$$\begin{aligned} d_1 &= V_0\theta_1^2 + (V_1 - 3V_0)\theta_1 + (V_2 - 2V_1 + V_0), \\ d_2 &= V_0\theta_2^2 + (V_1 - 3V_0)\theta_2 + (V_2 - 2V_1 + V_0), \\ d_3 &= V_0\theta_3^2 + (V_1 - 3V_0)\theta_3 + (V_2 - 2V_1 + V_0). \end{aligned}$$

Proof. By using Lemma 3.1, the proof follows from Theorem 1.2. \square

Note that from (2.2) and (3.2), we have

$$\begin{aligned} b_2 - (\theta_2 + \theta_3)b_1 + \theta_2\theta_3b_0 &= V_0\theta_1^2 + (V_1 - 3V_0)\theta_1 + (V_2 - 2V_1 + V_0), \\ b_2 - (\theta_1 + \theta_3)b_1 + \theta_1\theta_3b_0 &= V_0\theta_2^2 + (V_1 - 3V_0)\theta_2 + (V_2 - 2V_1 + V_0), \\ b_2 - (\theta_1 + \theta_2)b_1 + \theta_1\theta_2b_0 &= V_0\theta_3^2 + (V_1 - 3V_0)\theta_3 + (V_2 - 2V_1 + V_0), \end{aligned}$$

or

$$\begin{aligned} (V_0 + 2V_1 + V_2) - (\theta_2 + \theta_3)(V_0 + V_1) + \theta_2\theta_3V_0 &= V_0\theta_1^2 + (V_1 - 3V_0)\theta_1 + (V_2 - 2V_1 + V_0), \\ (V_0 + 2V_1 + V_2) - (\theta_1 + \theta_3)(V_0 + V_1) + \theta_1\theta_3V_0 &= V_0\theta_2^2 + (V_1 - 3V_0)\theta_2 + (V_2 - 2V_1 + V_0), \\ (V_0 + 2V_1 + V_2) - (\theta_1 + \theta_2)(V_0 + V_1) + \theta_1\theta_2V_0 &= V_0\theta_3^2 + (V_1 - 3V_0)\theta_3 + (V_2 - 2V_1 + V_0). \end{aligned}$$

Next, using Theorem 3.3, we present the Binet's formulas of binomial transform of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences.

Corollary 3.4. Binet's formulas of binomial transform of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences are

$$\begin{aligned} \widehat{T}_n &= \frac{(-1 + \theta_1)\theta_1^n}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{(-1 + \theta_2)\theta_2^n}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{(-1 + \theta_3)\theta_3^n}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{K}_n &= \theta_1^n + \theta_2^n + \theta_3^n, \\ \widehat{M}_n &= \frac{(3\theta_1^2 - 7\theta_1 + 6)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{(3\theta_2^2 - 7\theta_2 + 6)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{(3\theta_3^2 - 7\theta_3 + 6)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{U}_n &= \frac{2(\theta_1 - 1)^2\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(\theta_2 - 1)^2\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(\theta_3 - 1)^2\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{G}_n &= \frac{2(4\theta_1^2 - 5\theta_1 + 4)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(4\theta_2^2 - 5\theta_2 + 4)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(4\theta_3^2 - 5\theta_3 + 4)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \\ \widehat{H}_n &= \frac{2(3\theta_1 - 2)(\theta_1 - 2)\theta_1^{n-1}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{2(3\theta_2 - 2)(\theta_2 - 2)\theta_2^{n-1}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{2(3\theta_3 - 2)(\theta_3 - 2)\theta_3^{n-1}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}, \end{aligned}$$

respectively.

4 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Tribonacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Tribonacci Numbers). *For all integers n , we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \quad (4.1)$$

Proof. (4.1) is given in Soykan [23, Theorem 3.1]. \square

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+3} = 4b_{n+2} - 4b_{n+1} + 2b_n$, $r = 4$, $s = -4$, $t = 2$, we have the following proposition.

Proposition 4.1. *For all integers n , Simson formula of binomial transforms of generalized Tribonacci numbers is given as*

$$\begin{vmatrix} b_{n+2} & b_{n+1} & b_n \\ b_{n+1} & b_n & b_{n-1} \\ b_n & b_{n-1} & b_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.$$

The previous proposition gives the following results as particular examples.

Corollary 4.2. *For all integers n , Simson formula of binomial transforms of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas numbers are given as*

$$\begin{aligned} \begin{vmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{vmatrix} &= -2^{n-2}, \\ \begin{vmatrix} \widehat{K}_{n+2} & \widehat{K}_{n+1} & \widehat{K}_n \\ \widehat{K}_{n+1} & \widehat{K}_n & \widehat{K}_{n-1} \\ \widehat{K}_n & \widehat{K}_{n-1} & \widehat{K}_{n-2} \end{vmatrix} &= -11 \times 2^n, \\ \begin{vmatrix} \widehat{M}_{n+2} & \widehat{M}_{n+1} & \widehat{M}_n \\ \widehat{M}_{n+1} & \widehat{M}_n & \widehat{M}_{n-1} \\ \widehat{M}_n & \widehat{M}_{n-1} & \widehat{M}_{n-2} \end{vmatrix} &= -41 \times 2^{n-2}, \\ \begin{vmatrix} \widehat{U}_{n+2} & \widehat{U}_{n+1} & \widehat{U}_n \\ \widehat{U}_{n+1} & \widehat{U}_n & \widehat{U}_{n-1} \\ \widehat{U}_n & \widehat{U}_{n-1} & \widehat{U}_{n-2} \end{vmatrix} &= -2^n, \\ \begin{vmatrix} \widehat{G}_{n+2} & \widehat{G}_{n+1} & \widehat{G}_n \\ \widehat{G}_{n+1} & \widehat{G}_n & \widehat{G}_{n-1} \\ \widehat{G}_n & \widehat{G}_{n-1} & \widehat{G}_{n-2} \end{vmatrix} &= -11 \times 2^{n+1}, \\ \begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} &= -11 \times 2^{n+2}, \end{aligned}$$

respectively.

5 Some Identities

In this section, we obtain some identities of Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas, adjusted Tribonacci-Lucas numbers. First, we can give a few basic relations between $\{\widehat{T}_n\}$ and $\{\widehat{K}_n\}$.

Lemma 5.1. *The following equalities are true:*

$$\begin{aligned} 44\widehat{T}_n &= -5\widehat{K}_{n+4} + 24\widehat{K}_{n+3} - 26\widehat{K}_{n+2}, \\ 22\widehat{T}_n &= 2\widehat{K}_{n+3} - 3\widehat{K}_{n+2} - 5\widehat{K}_{n+1}, \\ 22\widehat{T}_n &= 5\widehat{K}_{n+2} - 13\widehat{K}_{n+1} + 4\widehat{K}_n, \\ 22\widehat{T}_n &= 7\widehat{K}_{n+1} - 16\widehat{K}_n + 10\widehat{K}_{n-1}, \\ 11\widehat{T}_n &= 6\widehat{K}_n - 9\widehat{K}_{n-1} + 7\widehat{K}_{n-2}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \widehat{K}_n &= -3\widehat{T}_{n+4} + 9\widehat{T}_{n+3} - \widehat{T}_{n+2}, \\ \widehat{K}_n &= -3\widehat{T}_{n+3} + 11\widehat{T}_{n+2} - 6\widehat{T}_{n+1}, \\ \widehat{K}_n &= -\widehat{T}_{n+2} + 6\widehat{T}_{n+1} - 6\widehat{T}_n, \\ \widehat{K}_n &= 2\widehat{T}_{n+1} - 2\widehat{T}_n - 2\widehat{T}_{n-1}, \\ \widehat{K}_n &= 6\widehat{T}_n - 10\widehat{T}_{n-1} + 4\widehat{T}_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n . We prove (5.1). To show (5.1) by writing

$$\widehat{T}_n = a \times \widehat{K}_{n+4} + b \times \widehat{K}_{n+3} + c \times \widehat{K}_{n+2}$$

and solving the system of equations

$$\begin{aligned} \widehat{T}_0 &= a \times \widehat{K}_4 + b \times \widehat{K}_3 + c \times \widehat{K}_2 \\ \widehat{T}_1 &= a \times \widehat{K}_5 + b \times \widehat{K}_4 + c \times \widehat{K}_3 \\ \widehat{T}_2 &= a \times \widehat{K}_6 + b \times \widehat{K}_5 + c \times \widehat{K}_4 \end{aligned}$$

we find that $a = -\frac{5}{44}, b = \frac{6}{11}, c = -\frac{13}{22}$. The other equalities can be proved similarly. \square

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{\widehat{T}_n\}$ and $\{\widehat{M}_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned} 82\widehat{T}_n &= -10\widehat{M}_{n+4} + 45\widehat{M}_{n+3} - 42\widehat{M}_{n+2}, \\ 82\widehat{T}_n &= 5\widehat{M}_{n+3} - 2\widehat{M}_{n+2} - 20\widehat{M}_{n+1}, \\ 41\widehat{T}_n &= 9\widehat{M}_{n+2} - 20\widehat{M}_{n+1} + 5\widehat{M}_n, \\ 41\widehat{T}_n &= 16\widehat{M}_{n+1} - 31\widehat{M}_n + 18\widehat{M}_{n-1}, \\ 41\widehat{T}_n &= 33\widehat{M}_n - 46\widehat{M}_{n-1} + 32\widehat{M}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 2\widehat{M}_n &= -8\widehat{T}_{n+4} + 25\widehat{T}_{n+3} - 6\widehat{T}_{n+2}, \\ 2\widehat{M}_n &= -7\widehat{T}_{n+3} + 26\widehat{T}_{n+2} - 16\widehat{T}_{n+1}, \\ \widehat{M}_n &= -\widehat{T}_{n+2} + 6\widehat{T}_{n+1} - 7\widehat{T}_n, \\ \widehat{M}_n &= 2\widehat{T}_{n+1} - 3\widehat{T}_n - 2\widehat{T}_{n-1}, \\ \widehat{M}_n &= 5\widehat{T}_n - 10\widehat{T}_{n-1} + 4\widehat{T}_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{T}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned} 4\widehat{T}_n &= \widehat{U}_{n+4} - 2\widehat{U}_{n+3} - 2\widehat{U}_{n+2}, \\ 2\widehat{T}_n &= \widehat{U}_{n+3} - 3\widehat{U}_{n+2} + \widehat{U}_{n+1}, \\ 2\widehat{T}_n &= \widehat{U}_{n+2} - 3\widehat{U}_{n+1} + 2\widehat{U}_n, \\ 2\widehat{T}_n &= \widehat{U}_{n+1} - 2\widehat{U}_n + 2\widehat{U}_{n-1}, \\ \widehat{T}_n &= \widehat{U}_n - \widehat{U}_{n-1} + \widehat{U}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{U}_n &= -\widehat{T}_{n+3} + 3\widehat{T}_{n+2}, \\ \widehat{U}_n &= -\widehat{T}_{n+2} + 4\widehat{T}_{n+1} - 2\widehat{T}_n, \\ \widehat{U}_n &= 2\widehat{T}_n - 2\widehat{T}_{n-1}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{T}_n\}$ and $\{\widehat{G}_n\}$.

Lemma 5.4. *The following equalities are true:*

$$\begin{aligned} 44\widehat{T}_n &= -13\widehat{G}_{n+4} + 47\widehat{G}_{n+3} - 28\widehat{G}_{n+2}, \\ 44\widehat{T}_n &= -5\widehat{G}_{n+3} + 24\widehat{G}_{n+2} - 26\widehat{G}_{n+1}, \\ 22\widehat{T}_n &= 2\widehat{G}_{n+2} - 3\widehat{G}_{n+1} - 5\widehat{G}_n, \\ 22\widehat{T}_n &= 5\widehat{G}_{n+1} - 13\widehat{G}_n + 4\widehat{G}_{n-1}, \\ 22\widehat{T}_n &= 7\widehat{G}_n - 16\widehat{G}_{n-1} + 10\widehat{G}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{G}_n &= -3\widehat{T}_{n+4} + 11\widehat{T}_{n+3} - 6\widehat{T}_{n+2}, \\ \widehat{G}_n &= -\widehat{T}_{n+3} + 6\widehat{T}_{n+2} - 6\widehat{T}_{n+1}, \\ \widehat{G}_n &= 2\widehat{T}_{n+2} - 2\widehat{T}_{n+1} - 2\widehat{T}_n, \\ \widehat{G}_n &= 6\widehat{T}_{n+1} - 10\widehat{T}_n + 4\widehat{T}_{n-1}, \\ \widehat{G}_n &= 14\widehat{T}_n - 20\widehat{T}_{n-1} + 12\widehat{T}_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{T}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.5. *The following equalities are true:*

$$\begin{aligned} 44\widehat{T}_n &= 2\widehat{H}_{n+4} - 3\widehat{H}_{n+3} - 5\widehat{H}_{n+2}, \\ 44\widehat{T}_n &= 5\widehat{H}_{n+3} - 13\widehat{H}_{n+2} + 4\widehat{H}_{n+1}, \\ 44\widehat{T}_n &= 7\widehat{H}_{n+2} - 16\widehat{H}_{n+1} + 10\widehat{H}_n, \\ 22\widehat{T}_n &= 6\widehat{H}_{n+1} - 9\widehat{H}_n + 7\widehat{H}_{n-1}, \\ 22\widehat{T}_n &= 15\widehat{H}_n - 17\widehat{H}_{n-1} + 12\widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{H}_n &= -\widehat{T}_{n+4} - 2\widehat{T}_{n+3} + 14\widehat{T}_{n+2}, \\ \widehat{H}_n &= -6\widehat{T}_{n+3} + 18\widehat{T}_{n+2} - 2\widehat{T}_{n+1}, \\ \widehat{H}_n &= -6\widehat{T}_{n+2} + 22\widehat{T}_{n+1} - 12\widehat{T}_n, \\ \widehat{H}_n &= -2\widehat{T}_{n+1} + 12\widehat{T}_n - 12\widehat{T}_{n-1}, \\ \widehat{H}_n &= 4\widehat{T}_n - 4\widehat{T}_{n-1} - 4\widehat{T}_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{K}_n\}$ and $\{\widehat{M}_n\}$.

Lemma 5.6. *The following equalities are true:*

$$\begin{aligned} 41\widehat{K}_n &= 36\widehat{M}_{n+4} - 121\widehat{M}_{n+3} + 61\widehat{M}_{n+2}, \\ 41\widehat{K}_n &= 23\widehat{M}_{n+3} - 83\widehat{M}_{n+2} + 72\widehat{M}_{n+1}, \\ 41\widehat{K}_n &= 9\widehat{M}_{n+2} - 20\widehat{M}_{n+1} + 46\widehat{M}_n, \\ 41\widehat{K}_n &= 16\widehat{M}_{n+1} + 10\widehat{M}_n + 18\widehat{M}_{n-1}, \\ 41\widehat{K}_n &= 74\widehat{M}_n - 46\widehat{M}_{n-1} + 32\widehat{M}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 44\widehat{M}_n &= 49\widehat{K}_{n+4} - 178\widehat{K}_{n+3} + 114\widehat{K}_{n+2}, \\ 22\widehat{M}_n &= 9\widehat{K}_{n+3} - 41\widehat{K}_{n+2} + 49\widehat{K}_{n+1}, \\ 22\widehat{M}_n &= -5\widehat{K}_{n+2} + 13\widehat{K}_{n+1} + 18\widehat{K}_n, \\ 22\widehat{M}_n &= -7\widehat{K}_{n+1} + 38\widehat{K}_n - 10\widehat{K}_{n-1}, \\ 11\widehat{M}_n &= 5\widehat{K}_n + 9\widehat{K}_{n-1} - 7\widehat{K}_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{U}_n\}$ and $\{\widehat{K}_n\}$.

Lemma 5.7. *The following equalities are true:*

$$\begin{aligned} 2\widehat{K}_n &= 2\widehat{U}_{n+4} - 9\widehat{U}_{n+3} + 10\widehat{U}_{n+2}, \\ 2\widehat{K}_n &= -\widehat{U}_{n+3} + 2\widehat{U}_{n+2} + 4\widehat{U}_{n+1}, \\ \widehat{K}_n &= -\widehat{U}_{n+2} + 4\widehat{U}_{n+1} - \widehat{U}_n, \\ \widehat{K}_n &= 3\widehat{U}_n - 2\widehat{U}_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 22\widehat{U}_n &= 8\widehat{K}_{n+4} - 23\widehat{K}_{n+3} + 2\widehat{K}_{n+2}, \\ 22\widehat{U}_n &= 9\widehat{K}_{n+3} - 30\widehat{K}_{n+2} + 16\widehat{K}_{n+1}, \\ 11\widehat{U}_n &= 3\widehat{K}_{n+2} - 10\widehat{K}_{n+1} + 9\widehat{K}_n, \\ 11\widehat{U}_n &= 2\widehat{K}_{n+1} - 3\widehat{K}_n + 6\widehat{K}_{n-1}, \\ 11\widehat{U}_n &= 5\widehat{K}_n - 2\widehat{K}_{n-1} + 4\widehat{K}_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{K}_n\}$ and $\{\widehat{G}_n\}$.

Lemma 5.8. *The following equalities are true:*

$$\begin{aligned} 2\widehat{K}_n &= 2\widehat{G}_{n+4} - 6\widehat{G}_{n+3} + \widehat{G}_{n+2}, \\ 2\widehat{K}_n &= 2\widehat{G}_{n+3} - 7\widehat{G}_{n+2} + 4\widehat{G}_{n+1}, \\ 2\widehat{K}_n &= \widehat{G}_{n+2} - 4\widehat{G}_{n+1} + 4\widehat{G}_n, \\ \widehat{K}_n &= \widehat{G}_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 2\widehat{G}_n &= \widehat{K}_{n+4} - 4\widehat{K}_{n+3} + 4\widehat{K}_{n+2}, \\ \widehat{G}_n &= \widehat{K}_{n+1}, \\ \widehat{G}_n &= 4\widehat{K}_n - 4\widehat{K}_{n-1} + 2\widehat{K}_{n-2}, \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{K}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.9. *The following equalities are true:*

$$\begin{aligned} 4\widehat{K}_n &= \widehat{H}_{n+4} - 4\widehat{H}_{n+3} + 4\widehat{H}_{n+2}, \\ 2\widehat{K}_n &= \widehat{H}_{n+1}, \\ \widehat{K}_n &= 2\widehat{H}_n - 2\widehat{H}_{n-1} + \widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{H}_n &= 2\widehat{K}_{n+4} - 6\widehat{K}_{n+3} + \widehat{K}_{n+2}, \\ \widehat{H}_n &= 2\widehat{K}_{n+3} - 7\widehat{K}_{n+2} + 4\widehat{K}_{n+1}, \\ \widehat{H}_n &= \widehat{K}_{n+2} - 4\widehat{K}_{n+1} + 4\widehat{K}_n, \\ \widehat{H}_n &= 2\widehat{K}_{n-1}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{M}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 5.10. *The following equalities are true:*

$$\begin{aligned} 4\widehat{M}_n &= 3\widehat{U}_{n+4} - 16\widehat{U}_{n+3} + 22\widehat{U}_{n+2}, \\ 2\widehat{M}_n &= -2\widehat{U}_{n+3} + 5\widehat{U}_{n+2} + 3\widehat{U}_{n+1}, \\ 2\widehat{M}_n &= -3\widehat{U}_{n+2} + 11\widehat{U}_{n+1} - 4\widehat{U}_n, \\ 2\widehat{M}_n &= -\widehat{U}_{n+1} + 8\widehat{U}_n - 6\widehat{U}_{n-1}, \\ \widehat{M}_n &= 2\widehat{U}_n - \widehat{U}_{n-1} - \widehat{U}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 41\widehat{U}_n &= 11\widehat{M}_{n+4} - 29\widehat{M}_{n+3} - 3\widehat{M}_{n+2}, \\ 41\widehat{U}_n &= 15\widehat{M}_{n+3} - 47\widehat{M}_{n+2} + 22\widehat{M}_{n+1}, \\ 41\widehat{U}_n &= 13\widehat{M}_{n+2} - 38\widehat{M}_{n+1} + 30\widehat{M}_n, \\ 41\widehat{U}_n &= 14\widehat{M}_{n+1} - 22\widehat{M}_n + 26\widehat{M}_{n-1}, \\ 41\widehat{U}_n &= 34\widehat{M}_n - 30\widehat{M}_{n-1} + 28\widehat{M}_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{M}_n\}$ and $\{\widehat{G}_n\}$.

Lemma 5.11. *The following equalities are true:*

$$\begin{aligned} 44\widehat{M}_n &= 57\widehat{G}_{n+4} - 179\widehat{G}_{n+3} + 50\widehat{G}_{n+2}, \\ 44\widehat{M}_n &= 49\widehat{G}_{n+3} - 178\widehat{G}_{n+2} + 114\widehat{G}_{n+1}, \\ 22\widehat{M}_n &= 9\widehat{G}_{n+2} - 41\widehat{G}_{n+1} + 49\widehat{G}_n, \\ 22\widehat{M}_n &= -5\widehat{G}_{n+1} + 13\widehat{G}_n + 18\widehat{G}_{n-1}, \\ 22\widehat{M}_n &= -7\widehat{G}_n + 38\widehat{G}_{n-1} - 10\widehat{G}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 41\widehat{G}_n &= 23\widehat{M}_{n+4} - 83\widehat{M}_{n+3} + 72\widehat{M}_{n+2}, \\ 41\widehat{G}_n &= 9\widehat{M}_{n+3} - 20\widehat{M}_{n+2} + 46\widehat{M}_{n+1}, \\ 41\widehat{G}_n &= 16\widehat{M}_{n+2} + 10\widehat{M}_{n+1} + 18\widehat{M}_n, \\ 41\widehat{G}_n &= 74\widehat{M}_{n+1} - 46\widehat{M}_n + 32\widehat{M}_{n-1}, \\ 41\widehat{G}_n &= 250\widehat{M}_n - 264\widehat{M}_{n-1} + 148\widehat{M}_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{M}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.12. *The following equalities are true:*

$$\begin{aligned} 44\widehat{M}_n &= 9\widehat{H}_{n+4} - 41\widehat{H}_{n+3} + 49\widehat{H}_{n+2}, \\ 44\widehat{M}_n &= -5\widehat{H}_{n+3} + 13\widehat{H}_{n+2} + 18\widehat{H}_{n+1}, \\ 44\widehat{M}_n &= -7\widehat{H}_{n+2} + 38\widehat{H}_{n+1} - 10\widehat{H}_n, \\ 22\widehat{M}_n &= 5\widehat{H}_{n+1} + 9\widehat{H}_n - 7\widehat{H}_{n-1}, \\ 22\widehat{M}_n &= 29\widehat{H}_n - 27\widehat{H}_{n-1} + 10\widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 41\widehat{H}_n &= 61\widehat{M}_{n+4} - 172\widehat{M}_{n+3} + 2\widehat{M}_{n+2}, \\ 41\widehat{H}_n &= 72\widehat{M}_{n+3} - 242\widehat{M}_{n+2} + 122\widehat{M}_{n+1}, \\ 41\widehat{H}_n &= 46\widehat{M}_{n+2} - 166\widehat{M}_{n+1} + 144\widehat{M}_n, \\ 41\widehat{H}_n &= 18\widehat{M}_{n+1} - 40\widehat{M}_n + 92\widehat{M}_{n-1}, \\ 41\widehat{H}_n &= 32\widehat{M}_n + 20\widehat{M}_{n-1} + 36\widehat{M}_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{U}_n\}$ and $\{\widehat{G}_n\}$.

Lemma 5.13. *The following equalities are true:*

$$\begin{aligned} 22\widehat{U}_n &= \widehat{G}_{n+4} + 4\widehat{G}_{n+3} - 19\widehat{G}_{n+2}, \\ 22\widehat{U}_n &= 8\widehat{G}_{n+3} - 23\widehat{G}_{n+2} + 2\widehat{G}_{n+1}, \\ 22\widehat{U}_n &= 9\widehat{G}_{n+2} - 30\widehat{G}_{n+1} + 16\widehat{G}_n, \\ 11\widehat{U}_n &= 3\widehat{G}_{n+1} - 10\widehat{G}_n + 9\widehat{G}_{n-1}, \\ 11\widehat{U}_n &= 2\widehat{G}_n - 3\widehat{G}_{n-1} + 6\widehat{G}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 2\widehat{G}_n &= -\widehat{U}_{n+4} + 2\widehat{U}_{n+3} + 4\widehat{U}_{n+2}, \\ \widehat{G}_n &= -\widehat{U}_{n+3} + 4\widehat{U}_{n+2} - \widehat{U}_{n+1}, \\ \widehat{G}_n &= 3\widehat{U}_{n+1} - 2\widehat{U}_n, \\ \widehat{G}_n &= 10\widehat{U}_n - 12\widehat{U}_{n-1} + 6\widehat{U}_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{\widehat{U}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.14. *The following equalities are true:*

$$\begin{aligned} 44\widehat{U}_n &= 9\widehat{H}_{n+4} - 30\widehat{H}_{n+3} + 16\widehat{H}_{n+2}, \\ 22\widehat{U}_n &= 3\widehat{H}_{n+3} - 10\widehat{H}_{n+2} + 9\widehat{H}_{n+1}, \\ 22\widehat{U}_n &= 2\widehat{H}_{n+2} - 3\widehat{H}_{n+1} + 6\widehat{H}_n, \\ 22\widehat{U}_n &= 5\widehat{H}_{n+1} - 2\widehat{H}_n + 4\widehat{H}_{n-1}, \\ 11\widehat{U}_n &= 9\widehat{H}_n - 8\widehat{H}_{n-1} + 5\widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{H}_n &= 5\widehat{U}_{n+4} - 18\widehat{U}_{n+3} + 11\widehat{U}_{n+2}, \\ \widehat{H}_n &= 2\widehat{U}_{n+3} - 9\widehat{U}_{n+2} + 10\widehat{U}_{n+1}, \\ \widehat{H}_n &= -\widehat{U}_{n+2} + 2\widehat{U}_{n+1} + 4\widehat{U}_n, \\ \widehat{H}_n &= -2\widehat{U}_{n+1} + 8\widehat{U}_n - 2\widehat{U}_{n-1}, \\ \widehat{H}_n &= 6\widehat{U}_{n-1} - 4\widehat{U}_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{\widehat{G}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.15. *The following equalities are true:*

$$\begin{aligned} 2\widehat{G}_n &= \widehat{H}_{n+2}, \\ \widehat{G}_n &= 2\widehat{H}_{n+1} - 2\widehat{H}_n + \widehat{H}_{n-1}, \\ \widehat{G}_n &= 6\widehat{H}_n - 7\widehat{H}_{n-1} + 4\widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 2\widehat{H}_n &= \widehat{G}_{n+4} - 8\widehat{G}_{n+2}, \\ \widehat{H}_n &= 2\widehat{G}_{n+3} - 6\widehat{G}_{n+2} + \widehat{G}_{n+1}, \\ \widehat{H}_n &= 2\widehat{G}_{n+2} - 7\widehat{G}_{n+1} + 4\widehat{G}_n, \\ \widehat{H}_n &= \widehat{G}_{n+1} - 4\widehat{G}_n + 4\widehat{G}_{n-1}, \\ \widehat{H}_n &= 2\widehat{G}_{n-2}. \end{aligned}$$

6 Sum Formulas

6.1 Sums of terms with positive subscripts

The following proposition presents some formulas of binomial transform of generalized Tribonacci numbers with positive subscripts.

Proposition 6.1. *If $r = 4, s = -4, t = 2$ then for $n \geq 0$, we have the following formulas:*

- (a) $\sum_{k=0}^n b_k = b_{n+3} - 3b_{n+2} + b_{n+1} - b_2 + 3b_1 - b_0$.
- (b) $\sum_{k=0}^n b_{2k} = \frac{1}{11}(5b_{2n+2} - 14b_{2n+1} + 12b_{2n} - 5b_2 + 14b_1 - b_0)$.
- (c) $\sum_{k=0}^n b_{2k+1} = \frac{1}{11}(6b_{2n+2} - 8b_{2n+1} + 10b_{2n} - 6b_2 + 19b_1 - 10b_0)$.

Proof. Take $r = 4, s = -4, t = 2$ in Theorem 2.1 in [24] (or take $x = 1, r = 4, s = -4, t = 2$ in Theorem 2.1 in [25]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tribonacci numbers (take $b_n = \widehat{T}_n$ with $\widehat{T}_0 = 0, \widehat{T}_1 = 1, \widehat{T}_2 = 3$).

Corollary 6.1. *For $n \geq 0$, we have the following formulas:*

- (a) $\sum_{k=0}^n \widehat{T}_k = \widehat{T}_{n+3} - 3\widehat{T}_{n+2} + \widehat{T}_{n+1}$.
- (b) $\sum_{k=0}^n \widehat{T}_{2k} = \frac{1}{11}(5\widehat{T}_{2n+2} - 14\widehat{T}_{2n+1} + 12\widehat{T}_{2n} - 1)$.
- (c) $\sum_{k=0}^n \widehat{T}_{2k+1} = \frac{1}{11}(6\widehat{T}_{2n+2} - 8\widehat{T}_{2n+1} + 10\widehat{T}_{2n} + 1)$.

Taking $b_n = \widehat{K}_n$ with $\widehat{K}_0 = 3, \widehat{K}_1 = 4, \widehat{K}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Tribonacci-Lucas numbers.

Corollary 6.2. *For $n \geq 0$, we have the following formulas:*

- (a) $\sum_{k=0}^n \widehat{K}_k = \widehat{K}_{n+3} - 3\widehat{K}_{n+2} + \widehat{K}_{n+1} + 1$.
- (b) $\sum_{k=0}^n \widehat{K}_{2k} = \frac{1}{11}(5\widehat{K}_{2n+2} - 14\widehat{K}_{2n+1} + 12\widehat{K}_{2n} + 13)$.
- (c) $\sum_{k=0}^n \widehat{K}_{2k+1} = \frac{1}{11}(6\widehat{K}_{2n+2} - 8\widehat{K}_{2n+1} + 10\widehat{K}_{2n} - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tribonacci-Perrin numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 3, \widehat{M}_1 = 3, \widehat{M}_2 = 5$).

Corollary 6.3. For $n \geq 0$, we have the following formulas:

- (a) $\sum_{k=0}^n \widehat{M}_k = \widehat{M}_{n+3} - 3\widehat{M}_{n+2} + \widehat{M}_{n+1} + 1.$
- (b) $\sum_{k=0}^n \widehat{M}_{2k} = \frac{1}{11}(5\widehat{M}_{2n+2} - 14\widehat{M}_{2n+1} + 12\widehat{M}_{2n} + 14).$
- (c) $\sum_{k=0}^n \widehat{M}_{2k+1} = \frac{1}{11}(6\widehat{M}_{2n+2} - 8\widehat{M}_{2n+1} + 10\widehat{M}_{2n} - 3).$

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 1, \widehat{U}_1 = 2, \widehat{U}_2 = 4$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified Tribonacci numbers.

Corollary 6.4. For $n \geq 0$, we have the following formulas:

- (a) $\sum_{k=0}^n \widehat{U}_k = \widehat{U}_{n+3} - 3\widehat{U}_{n+2} + \widehat{U}_{n+1} + 1.$
- (b) $\sum_{k=0}^n \widehat{U}_{2k} = \frac{1}{11}(5\widehat{U}_{2n+2} - 14\widehat{U}_{2n+1} + 12\widehat{U}_{2n} + 7).$
- (c) $\sum_{k=0}^n \widehat{U}_{2k+1} = \frac{1}{11}(6\widehat{U}_{2n+2} - 8\widehat{U}_{2n+1} + 10\widehat{U}_{2n} + 4).$

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of modified Tribonacci-Lucas numbers (take $b_n = \widehat{G}_n$ with $\widehat{G}_0 = 4, \widehat{G}_1 = 8, \widehat{G}_2 = 22$).

Corollary 6.5. For $n \geq 0$, binomial transform of modified Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n \widehat{G}_k = \widehat{G}_{n+3} - 3\widehat{G}_{n+2} + \widehat{G}_{n+1} - 2.$
- (b) $\sum_{k=0}^n \widehat{G}_{2k} = \frac{1}{11}(5\widehat{G}_{2n+2} - 14\widehat{G}_{2n+1} + 12\widehat{G}_{2n} - 2).$
- (c) $\sum_{k=0}^n \widehat{G}_{2k+1} = \frac{1}{11}(6\widehat{G}_{2n+2} - 8\widehat{G}_{2n+1} + 10\widehat{G}_{2n} - 20).$

Taking $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 4, \widehat{H}_1 = 6, \widehat{H}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of adjusted Tribonacci-Lucas numbers.

Corollary 6.6. For $n \geq 0$, binomial transform of adjusted Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n \widehat{H}_k = \widehat{H}_{n+3} - 3\widehat{H}_{n+2} + \widehat{H}_{n+1} + 6.$
- (b) $\sum_{k=0}^n \widehat{H}_{2k} = \frac{1}{11}(5\widehat{H}_{2n+2} - 14\widehat{H}_{2n+1} + 12\widehat{H}_{2n} + 40).$
- (c) $\sum_{k=0}^n \widehat{H}_{2k+1} = \frac{1}{11}(6\widehat{H}_{2n+2} - 8\widehat{H}_{2n+1} + 10\widehat{H}_{2n} + 26).$

6.2 Sums of terms with negative subscripts

The following proposition presents some formulas of binomial transform of generalized Tribonacci numbers with negative subscripts.

Proposition 6.2. If $r = 4, s = -4, t = 2$ then for $n \geq 1$ we have the following formulas:

- (a) $\sum_{k=1}^n b_{-k} = -2b_{-n-1} + 2b_{-n-2} - 2b_{-n-3} + b_2 - 3b_1 + b_0.$
- (b) $\sum_{k=1}^n b_{-2k} = \frac{1}{11}(-6b_{-2n+1} + 19b_{-2n} - 10b_{-2n-1} + 5b_2 - 14b_1 + b_0).$
- (c) $\sum_{k=1}^n b_{-2k+1} = \frac{1}{11}(-5b_{-2n+1} + 14b_{-2n} - 12b_{-2n-1} + 6b_2 - 19b_1 + 10b_0).$

Proof. Take $r = 4, s = -4, t = 2$ in Theorem 3.1 in [24] or (or take $x = 1, r = 4, s = -4, t = 2$ in Theorem 3.1 in [25]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tribonacci numbers (take $b_n = \widehat{T}_n$ with $\widehat{T}_0 = 0, \widehat{T}_1 = 1, \widehat{T}_2 = 3$).

Corollary 6.7. For $n \geq 1$, binomial transform of Tribonacci numbers have the following properties.

- (a) $\sum_{k=1}^n \widehat{T}_{-k} = -2\widehat{T}_{-n-1} + 2\widehat{T}_{-n-2} - 2\widehat{T}_{-n-3}$.
- (b) $\sum_{k=1}^n \widehat{T}_{-2k} = \frac{1}{11}(-6\widehat{T}_{-2n+1} + 19\widehat{T}_{-2n} - 10\widehat{T}_{-2n-1} + 1)$.
- (c) $\sum_{k=1}^n \widehat{T}_{-2k+1} = \frac{1}{11}(-5\widehat{T}_{-2n+1} + 14\widehat{T}_{-2n} - 12\widehat{T}_{-2n-1} - 1)$.

Taking $b_n = \widehat{K}_n$ with $\widehat{K}_0 = 3, \widehat{K}_1 = 4, \widehat{K}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Tribonacci-Lucas numbers.

Corollary 6.8. For $n \geq 1$, binomial transform of Tribonacci-Lucas numbers have the following properties.

- (a) $\sum_{k=1}^n \widehat{K}_{-k} = -2\widehat{K}_{-n-1} + 2\widehat{K}_{-n-2} - 2\widehat{K}_{-n-3} - 1$.
- (b) $\sum_{k=1}^n \widehat{K}_{-2k} = \frac{1}{11}(-6\widehat{K}_{-2n+1} + 19\widehat{K}_{-2n} - 10\widehat{K}_{-2n-1} - 13)$.
- (c) $\sum_{k=1}^n \widehat{K}_{-2k+1} = \frac{1}{11}(-5\widehat{K}_{-2n+1} + 14\widehat{K}_{-2n} - 12\widehat{K}_{-2n-1} + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tribonacci-Perrin numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 3, \widehat{M}_1 = 3, \widehat{M}_2 = 5$).

Corollary 6.9. For $n \geq 1$, binomial transform of Tribonacci-Perrin numbers have the following properties.

- (a) $\sum_{k=1}^n \widehat{M}_{-k} = -2\widehat{M}_{-n-1} + 2\widehat{M}_{-n-2} - 2\widehat{M}_{-n-3} - 1$.
- (b) $\sum_{k=1}^n \widehat{M}_{-2k} = \frac{1}{11}(-6\widehat{M}_{-2n+1} + 19\widehat{M}_{-2n} - 10\widehat{M}_{-2n-1} - 14)$.
- (c) $\sum_{k=1}^n \widehat{M}_{-2k+1} = \frac{1}{11}(-5\widehat{M}_{-2n+1} + 14\widehat{M}_{-2n} - 12\widehat{M}_{-2n-1} + 3)$.

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 1, \widehat{U}_1 = 2, \widehat{U}_2 = 4$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified Tribonacci numbers.

Corollary 6.10. For $n \geq 1$, binomial transform of modified Tribonacci numbers have the following properties.

- (a) $\sum_{k=1}^n \widehat{U}_{-k} = -2\widehat{U}_{-n-1} + 2\widehat{U}_{-n-2} - 2\widehat{U}_{-n-3} - 1$.
- (b) $\sum_{k=1}^n \widehat{U}_{-2k} = \frac{1}{11}(-6\widehat{U}_{-2n+1} + 19\widehat{U}_{-2n} - 10\widehat{U}_{-2n-1} - 7)$.
- (c) $\sum_{k=1}^n \widehat{U}_{-2k+1} = \frac{1}{11}(-5\widehat{U}_{-2n+1} + 14\widehat{U}_{-2n} - 12\widehat{U}_{-2n-1} - 4)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of modified Tribonacci-Lucas numbers (take $b_n = \widehat{G}_n$ with $\widehat{G}_0 = 4, \widehat{G}_1 = 8, \widehat{G}_2 = 22$).

Corollary 6.11. For $n \geq 1$, binomial transform of modified Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=1}^n \widehat{G}_{-k} = -2\widehat{G}_{-n-1} + 2\widehat{G}_{-n-2} - 2\widehat{G}_{-n-3} + 2$.
- (b) $\sum_{k=1}^n \widehat{G}_{-2k} = \frac{1}{11}(-6\widehat{G}_{-2n+1} + 19\widehat{G}_{-2n} - 10\widehat{G}_{-2n-1} + 2)$.
- (c) $\sum_{k=1}^n \widehat{G}_{-2k+1} = \frac{1}{11}(-5\widehat{G}_{-2n+1} + 14\widehat{G}_{-2n} - 12\widehat{G}_{-2n-1} + 20)$.

Taking $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 4, \widehat{H}_1 = 6, \widehat{H}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of adjusted Tribonacci-Lucas numbers.

Corollary 6.12. For $n \geq 1$, binomial transform of adjusted Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=1}^n \widehat{H}_{-k} = -2\widehat{H}_{-n-1} + 2\widehat{H}_{-n-2} - 2\widehat{H}_{-n-3} - 6$.
- (b) $\sum_{k=1}^n \widehat{H}_{-2k} = \frac{1}{11}(-6\widehat{H}_{-2n+1} + 19\widehat{H}_{-2n} - 10\widehat{H}_{-2n-1} - 40)$.
- (c) $\sum_{k=1}^n \widehat{H}_{-2k+1} = \frac{1}{11}(-5\widehat{H}_{-2n+1} + 14\widehat{H}_{-2n} - 12\widehat{H}_{-2n-1} - 26)$.

6.3 Sums of squares of terms with positive subscripts

The following proposition presents some formulas of binomial transform of generalized Tribonacci numbers with positive subscripts.

Proposition 6.3. *If $r = 4, s = -4, t = 2$ then for $n \geq 0$ we have the following formulas:*

- (a) $\sum_{k=0}^n b_k^2 = \frac{1}{11}(-7b_{n+3}^2 - 87b_{n+2}^2 - 39b_{n+1}^2 + 48b_{n+3}b_{n+2} - 16b_{n+3}b_{n+1} + 80b_{n+2}b_{n+1} + 7b_2^2 + 87b_1^2 + 39b_0^2 - 48b_2b_1 + 16b_2b_0 - 80b_1b_0).$
- (b) $\sum_{k=0}^n b_{k+1}b_k = \frac{1}{11}(-4b_{n+3}^2 - 56b_{n+2}^2 - 16b_{n+1}^2 + 29b_{n+3}b_{n+2} - 6b_{n+3}b_{n+1} + 41b_{n+2}b_{n+1} + 4b_2^2 + 56b_1^2 + 16b_0^2 - 29b_2b_1 + 6b_2b_0 - 41b_1b_0).$
- (c) $\sum_{k=0}^n b_{k+2}b_k = \frac{1}{11}(4b_{n+3}^2 + 12b_{n+2}^2 + 16b_{n+1}^2 - 18b_{n+3}b_{n+2} + 17b_{n+3}b_{n+1} - 30b_{n+2}b_{n+1} - 4b_2^2 - 12b_1^2 - 16b_0^2 + 18b_2b_1 - 17b_2b_0 + 30b_1b_0).$

Proof. Take $x = 1, r = 4, s = -4, t = 2$ in Theorem 4.1 in [26], see also [27].

From the last proposition, we have the following Corollary which gives sum formulas of binomial transform of Tribonacci numbers (take $b_n = \widehat{T}_n$ with $\widehat{T}_0 = 0, \widehat{T}_1 = 1, \widehat{T}_2 = 3$).

Corollary 6.13. *For $n \geq 0$, binomial transform of Tribonacci numbers have the following properties:*

- (a) $\sum_{k=0}^n \widehat{T}_k^2 = \frac{1}{11}(-7\widehat{T}_{n+3}^2 - 87\widehat{T}_{n+2}^2 - 39\widehat{T}_{n+1}^2 + 48\widehat{T}_{n+3}\widehat{T}_{n+2} - 16\widehat{T}_{n+3}\widehat{T}_{n+1} + 80\widehat{T}_{n+2}\widehat{T}_{n+1} + 6).$
- (b) $\sum_{k=0}^n \widehat{T}_{k+1}\widehat{T}_k = \frac{1}{11}(-4\widehat{T}_{n+3}^2 - 56\widehat{T}_{n+2}^2 - 16\widehat{T}_{n+1}^2 + 29\widehat{T}_{n+3}\widehat{T}_{n+2} - 6\widehat{T}_{n+3}\widehat{T}_{n+1} + 41\widehat{T}_{n+2}\widehat{T}_{n+1} + 5).$
- (c) $\sum_{k=0}^n \widehat{T}_{k+2}\widehat{T}_k = \frac{1}{11}(4\widehat{T}_{n+3}^2 + 12\widehat{T}_{n+2}^2 + 16\widehat{T}_{n+1}^2 - 18\widehat{T}_{n+3}\widehat{T}_{n+2} + 17\widehat{T}_{n+3}\widehat{T}_{n+1} - 30\widehat{T}_{n+2}\widehat{T}_{n+1} + 6).$

Taking $b_n = \widehat{K}_n$ with $\widehat{K}_0 = 3, \widehat{K}_1 = 4, \widehat{K}_2 = 8$ in the last Proposition, we have the following Corollary which presents sum formulas of binomial transform of Tribonacci-Lucas numbers.

Corollary 6.14. *For $n \geq 0$, binomial transform of Tribonacci-Lucas numbers have the following properties:*

- (a) $\sum_{k=0}^n \widehat{K}_k^2 = \frac{1}{11}(-7\widehat{K}_{n+3}^2 - 87\widehat{K}_{n+2}^2 - 39\widehat{K}_{n+1}^2 + 48\widehat{K}_{n+3}\widehat{K}_{n+2} - 16\widehat{K}_{n+3}\widehat{K}_{n+1} + 80\widehat{K}_{n+2}\widehat{K}_{n+1} + 79).$
- (b) $\sum_{k=0}^n \widehat{K}_{k+1}\widehat{K}_k = \frac{1}{11}(-4\widehat{K}_{n+3}^2 - 56\widehat{K}_{n+2}^2 - 16\widehat{K}_{n+1}^2 + 29\widehat{K}_{n+3}\widehat{K}_{n+2} - 6\widehat{K}_{n+3}\widehat{K}_{n+1} + 41\widehat{K}_{n+2}\widehat{K}_{n+1} + 20).$
- (c) $\sum_{k=0}^n \widehat{K}_{k+2}\widehat{K}_k = \frac{1}{11}(4\widehat{K}_{n+3}^2 + 12\widehat{K}_{n+2}^2 + 16\widehat{K}_{n+1}^2 - 18\widehat{K}_{n+3}\widehat{K}_{n+2} + 17\widehat{K}_{n+3}\widehat{K}_{n+1} - 30\widehat{K}_{n+2}\widehat{K}_{n+1} - 64).$

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tribonacci-Perrin numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 3, \widehat{M}_1 = 3, \widehat{M}_2 = 5$).

Corollary 6.15. *For $n \geq 0$, binomial transform of Tribonacci-Perrin numbers have the following properties:*

- (a) $\sum_{k=0}^n \widehat{M}_k^2 = \frac{1}{11}(-7\widehat{M}_{n+3}^2 - 87\widehat{M}_{n+2}^2 - 39\widehat{M}_{n+1}^2 + 48\widehat{M}_{n+3}\widehat{M}_{n+2} - 16\widehat{M}_{n+3}\widehat{M}_{n+1} + 80\widehat{M}_{n+2}\widehat{M}_{n+1} + 109).$
- (b) $\sum_{k=0}^n \widehat{M}_{k+1}\widehat{M}_k = \frac{1}{11}(-4\widehat{M}_{n+3}^2 - 56\widehat{M}_{n+2}^2 - 16\widehat{M}_{n+1}^2 + 29\widehat{M}_{n+3}\widehat{M}_{n+2} - 6\widehat{M}_{n+3}\widehat{M}_{n+1} + 41\widehat{M}_{n+2}\widehat{M}_{n+1} + 34).$
- (c) $\sum_{k=0}^n \widehat{M}_{k+2}\widehat{M}_k = \frac{1}{11}(4\widehat{M}_{n+3}^2 + 12\widehat{M}_{n+2}^2 + 16\widehat{M}_{n+1}^2 - 18\widehat{M}_{n+3}\widehat{M}_{n+2} + 17\widehat{M}_{n+3}\widehat{M}_{n+1} - 30\widehat{M}_{n+2}\widehat{M}_{n+1} - 67).$

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 1, \widehat{U}_1 = 2, \widehat{U}_2 = 4$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified Tribonacci numbers.

Corollary 6.16. For $n \geq 0$, binomial transform of modified Tribonacci numbers have the following properties:

- (a) $\sum_{k=0}^n \widehat{U}_k^2 = \frac{1}{11}(-7\widehat{U}_{n+3}^2 - 87\widehat{U}_{n+2}^2 - 39\widehat{U}_{n+1}^2 + 48\widehat{U}_{n+3}\widehat{U}_{n+2} - 16\widehat{U}_{n+3}\widehat{U}_{n+1} + 80\widehat{U}_{n+2}\widehat{U}_{n+1} + 19)$.
- (b) $\sum_{k=0}^n \widehat{U}_{k+1}\widehat{U}_k = \frac{1}{11}(-4\widehat{U}_{n+3}^2 - 56\widehat{U}_{n+2}^2 - 16\widehat{U}_{n+1}^2 + 29\widehat{U}_{n+3}\widehat{U}_{n+2} - 6\widehat{U}_{n+3}\widehat{U}_{n+1} + 41\widehat{U}_{n+2}\widehat{U}_{n+1} + 14)$.
- (c) $\sum_{k=0}^n \widehat{U}_{k+2}\widehat{U}_k = \frac{1}{11}(4\widehat{U}_{n+3}^2 + 12\widehat{U}_{n+2}^2 + 16\widehat{U}_{n+1}^2 - 18\widehat{U}_{n+3}\widehat{U}_{n+2} + 17\widehat{U}_{n+3}\widehat{U}_{n+1} - 30\widehat{U}_{n+2}\widehat{U}_{n+1} + 8)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of modified Tribonacci-Lucas numbers (take $b_n = \widehat{G}_n$ with $\widehat{G}_0 = 4, \widehat{G}_1 = 8, \widehat{G}_2 = 22$).

Corollary 6.17. For $n \geq 0$, binomial transform of modified Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n \widehat{G}_k^2 = \frac{1}{11}(-7\widehat{G}_{n+3}^2 - 87\widehat{G}_{n+2}^2 - 39\widehat{G}_{n+1}^2 + 48\widehat{G}_{n+3}\widehat{G}_{n+2} - 16\widehat{G}_{n+3}\widehat{G}_{n+1} + 80\widehat{G}_{n+2}\widehat{G}_{n+1} - 20)$.
- (b) $\sum_{k=0}^n \widehat{G}_{k+1}\widehat{G}_k = \frac{1}{11}(-4\widehat{G}_{n+3}^2 - 56\widehat{G}_{n+2}^2 - 16\widehat{G}_{n+1}^2 + 29\widehat{G}_{n+3}\widehat{G}_{n+2} - 6\widehat{G}_{n+3}\widehat{G}_{n+1} + 41\widehat{G}_{n+2}\widehat{G}_{n+1} - 112)$.
- (c) $\sum_{k=0}^n \widehat{G}_{k+2}\widehat{G}_k = \frac{1}{11}(4\widehat{G}_{n+3}^2 + 12\widehat{G}_{n+2}^2 + 16\widehat{G}_{n+1}^2 - 18\widehat{G}_{n+3}\widehat{G}_{n+2} + 17\widehat{G}_{n+3}\widehat{G}_{n+1} - 30\widehat{G}_{n+2}\widehat{G}_{n+1} - 328)$.

Taking $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 4, \widehat{H}_1 = 6, \widehat{H}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of adjusted Tribonacci-Lucas numbers.

Corollary 6.18. For $n \geq 0$, binomial transform of adjusted Tribonacci-Lucas numbers have the following properties:

- (a) $\sum_{k=0}^n \widehat{H}_k^2 = \frac{1}{11}(-7\widehat{H}_{n+3}^2 - 87\widehat{H}_{n+2}^2 - 39\widehat{H}_{n+1}^2 + 48\widehat{H}_{n+3}\widehat{H}_{n+2} - 16\widehat{H}_{n+3}\widehat{H}_{n+1} + 80\widehat{H}_{n+2}\widehat{H}_{n+1} + 492)$.
- (b) $\sum_{k=0}^n \widehat{H}_{k+1}\widehat{H}_k = \frac{1}{11}(-4\widehat{H}_{n+3}^2 - 56\widehat{H}_{n+2}^2 - 16\widehat{H}_{n+1}^2 + 29\widehat{H}_{n+3}\widehat{H}_{n+2} - 6\widehat{H}_{n+3}\widehat{H}_{n+1} + 41\widehat{H}_{n+2}\widehat{H}_{n+1} + 344)$.
- (c) $\sum_{k=0}^n \widehat{H}_{k+2}\widehat{H}_k = \frac{1}{11}(4\widehat{H}_{n+3}^2 + 12\widehat{H}_{n+2}^2 + 16\widehat{H}_{n+1}^2 - 18\widehat{H}_{n+3}\widehat{H}_{n+2} + 17\widehat{H}_{n+3}\widehat{H}_{n+1} - 30\widehat{H}_{n+2}\widehat{H}_{n+1} + 96)$.

7 Matrices Related with Binomial Transform of Generalized Tribonacci Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [28]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (2.1) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix} \quad (7.2)$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n = \widehat{T}_n$ in (7.2) we have

$$\begin{pmatrix} \widehat{T}_{n+2} \\ \widehat{T}_{n+1} \\ \widehat{T}_n \end{pmatrix} = \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{T}_{n+1} \\ \widehat{T}_n \\ \widehat{T}_{n-1} \end{pmatrix}. \quad (7.3)$$

For $n \geq 0$, we define

$$B_n = \begin{pmatrix} \sum_{k=0}^{n+1} \widehat{T}_k & -2(2\sum_{k=0}^n \widehat{T}_k - \sum_{k=0}^{n-1} \widehat{T}_k) & 2\sum_{k=0}^n \widehat{T}_k \\ \sum_{k=0}^n \widehat{T}_k & -2(2\sum_{k=0}^{n-1} \widehat{T}_k - \sum_{k=0}^{n-2} \widehat{T}_k) & 2\sum_{k=0}^{n-1} \widehat{T}_k \\ \sum_{k=0}^{n-1} \widehat{T}_k & -2(2\sum_{k=0}^{n-2} \widehat{T}_k - \sum_{k=0}^{n-3} \widehat{T}_k) & 2\sum_{k=0}^{n-2} \widehat{T}_k \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -4b_n + 2b_{n-1} & 2b_n \\ b_n & -4b_{n-1} + 2b_{n-2} & 2b_{n-1} \\ b_{n-1} & -4b_{n-2} + 2b_{n-3} & 2b_{n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{T}_k = 0, \quad \sum_{k=0}^{-2} \widehat{T}_k = \frac{1}{2}, \quad \sum_{k=0}^{-3} \widehat{T}_k = 1.$$

Theorem 7.1. For all integers $m, n \geq 0$, we have

- (a) $B_n = A^n$.
- (b) $C_1 A^n = A^n C_1$.
- (c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

- (a) Proof can be done by mathematical induction on n .
- (b) After matrix multiplication, (b) follows.
- (c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -4b_{n-1} + 2b_{n-2} & 2b_{n-1} \\ b_{n-1} & -4b_{n-2} + 2b_{n-3} & 2b_{n-2} \\ b_{n-2} & -4b_{n-3} + 2b_{n-4} & 2b_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{n+1} & -4b_n + 2b_{n-1} & 2b_n \\ b_n & -4b_{n-1} + 2b_{n-2} & 2b_{n-1} \\ b_{n-1} & -4b_{n-2} + 2b_{n-3} & 2b_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Some properties of matrix A^n can be given as

$$A^n = 4A^{n-1} - 4A^{n-2} + 2A^{n-3} = 2A^{n+1} - 2A^{n+2} + \frac{1}{2}A^{n+3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 2^n$$

for all integers $m, n \geq 0$.

Theorem 7.2. For $m, n \geq 0$, we have

$$b_{n+m} = b_n \sum_{k=0}^{m+1} \hat{T}_k + b_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2b_{n-2} \sum_{k=0}^m \hat{T}_k \quad (7.4)$$

$$= b_n \sum_{k=0}^{m+1} \hat{T}_k + (-4b_{n-1} + 2b_{n-2}) \sum_{k=0}^m \hat{T}_k + 2b_{n-1} \sum_{k=0}^{m-1} \hat{T}_k. \quad (7.5)$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m . From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m . We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Corollary 7.3. For $m, n \geq 0$, we have

$$\begin{aligned} \hat{T}_{n+m} &= \hat{T}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{T}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{T}_{n-2} \sum_{k=0}^m \hat{T}_k, \\ \hat{K}_{n+m} &= \hat{K}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{K}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{K}_{n-2} \sum_{k=0}^m \hat{T}_k, \\ \hat{M}_{n+m} &= \hat{M}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{M}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{M}_{n-2} \sum_{k=0}^m \hat{T}_k, \\ \hat{U}_{n+m} &= \hat{U}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{U}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{U}_{n-2} \sum_{k=0}^m \hat{T}_k, \\ \hat{G}_{n+m} &= \hat{G}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{G}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{G}_{n-2} \sum_{k=0}^m \hat{T}_k, \\ \hat{H}_{n+m} &= \hat{H}_n \sum_{k=0}^{m+1} \hat{T}_k + \hat{H}_{n-1} \left(-4 \sum_{k=0}^m \hat{T}_k + 2 \sum_{k=0}^{m-1} \hat{T}_k \right) + 2\hat{H}_{n-2} \sum_{k=0}^m \hat{T}_k. \end{aligned}$$

From Corollary 6.1, we know that for $n \geq 0$,

$$\sum_{k=0}^n \hat{T}_k = \hat{T}_{n+3} - 3\hat{T}_{n+2} + \hat{T}_{n+1}.$$

So, Theorem 7.2 and Corollary 7.3 can be written in the following forms:

Theorem 7.4. For $m, n \geq 0$, we have

$$b_{n+m} = b_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + b_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) + b_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}). \quad (7.6)$$

Remark 7.1. By induction, it can be proved that for all integers $m, n \leq 0$, (7.6) holds. So, for all integers m, n , (7.6) is true.

Corollary 7.5. For all integers m, n , we have

$$\begin{aligned} \widehat{T}_{n+m} &= \widehat{T}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{T}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{T}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}), \\ \widehat{K}_{n+m} &= \widehat{K}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{K}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{K}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}), \\ \widehat{M}_{n+m} &= \widehat{M}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{M}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{M}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}), \\ \widehat{U}_{n+m} &= \widehat{U}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{U}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{U}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}), \\ \widehat{G}_{n+m} &= \widehat{G}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{G}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{G}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}), \\ \widehat{H}_{n+m} &= \widehat{H}_n(\widehat{T}_{m+4} - 3\widehat{T}_{m+3} + \widehat{T}_{m+2}) + \widehat{H}_{n-1}(-4\widehat{T}_{m+3} + 14\widehat{T}_{m+2} - 10\widehat{T}_{m+1} + 2\widehat{T}_m) \\ &\quad + \widehat{H}_{n-2}(2\widehat{T}_{m+3} - 6\widehat{T}_{m+2} + 2\widehat{T}_{m+1}). \end{aligned}$$

Now, we consider non-positive subscript cases. For $n \geq 0$, we define

$$B_{-n} = \begin{pmatrix} -\sum_{k=0}^{n-2} \widehat{T}_{-k} & 2(2\sum_{k=0}^{n-1} \widehat{T}_{-k} - \sum_{k=0}^n \widehat{T}_{-k}) & -2\sum_{k=0}^{n-1} \widehat{T}_{-k} \\ -\sum_{k=0}^{n-1} \widehat{T}_{-k} & 2(2\sum_{k=0}^n \widehat{T}_{-k} - \sum_{k=0}^{n+1} \widehat{T}_{-k}) & -2\sum_{k=0}^n \widehat{T}_{-k} \\ -\sum_{k=0}^n \widehat{T}_{-k} & 2(2\sum_{k=0}^{n+1} \widehat{T}_{-k} - \sum_{k=0}^{n+2} \widehat{T}_{-k}) & -2\sum_{k=0}^{n+1} \widehat{T}_{-k} \end{pmatrix}$$

and

$$C_{-n} = \begin{pmatrix} b_{-n+1} & -4b_{-n} + 2b_{-n-1} & 2b_{-n} \\ b_{-n} & -4b_{-n-1} + 2b_{-n-2} & 2b_{-n-1} \\ b_{-n-1} & -4b_{-n-2} + 2b_{-n-3} & 2b_{-n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{T}_{-k} = 0, \quad \sum_{k=0}^{-2} \widehat{T}_{-k} = -1.$$

Theorem 7.6. For all integers $m, n \geq 0$, we have

- (a) $B_{-n} = A^{-n}$.
- (b) $C_{-1}A^{-n} = A^{-n}C_{-1}$.
- (c) $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$.

Proof.

- (a) Proof can be done by mathematical induction on n .
- (b) After matrix multiplication, (b) follows.

(c) We have

$$\begin{aligned} A^{-1}C_{-n-1} &= \begin{pmatrix} 4 & -4 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{-n} & -4b_{-n-1} + 2b_{-n-2} & 2b_{-n-1} \\ b_{-n-1} & -4b_{-n-2} + 2b_{-n-3} & 2b_{-n-2} \\ b_{-n-2} & -4b_{-n-3} + 2b_{-n-4} & 2b_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{-n+1} & -4b_{-n} + 3b_{-n-1} & 2b_{-n} \\ b_{-n} & -4b_{-n-1} + 3b_{-n-2} & 2b_{-n-1} \\ b_{-n-1} & -4b_{-n-2} + 3b_{-n-3} & 2b_{-n-2} \end{pmatrix} = C_{-n}, \end{aligned}$$

i.e. $C_{-n} = A^{-1}C_{-n-1}$. From the last equation, using induction, we obtain $C_{-n} = A^{-n-1}C_{-1}$. Now,

$$C_{-n-m} = A^{-n-m-1}C_{-1} = A^{-n-1}A^{-m}C_{-1} = A^{-n-1}C_{-1}A^{-m} = C_{-n}B_{-m}$$

and similarly,

$$C_{-n-m} = B_{-m}C_{-n}.$$

□

Some properties of matrix A^{-n} can be given as

$$A^{-n} = 4A^{-n-1} - 4A^{-n-2} + 2A^{-n-3} = 2A^{-n+1} - 2A^{-n+2} + \frac{1}{2}A^{-n+3}$$

and

$$A^{-n-m} = A^{-n}A^{-m} = A^{-m}A^{-n}$$

and

$$\det(A^{-n}) = 2^{-n}$$

for all integers $m, n \geq 0$.

Theorem 7.7. For $m, n \geq 0$, we have

$$\begin{aligned} b_{-n-m} &= -b_{-n} \sum_{k=0}^{m-2} \hat{T}_{-k} - b_{-n-1} \left(-4 \sum_{k=0}^{m-1} \hat{T}_{-k} + 2 \sum_{k=0}^m \hat{T}_{-k} \right) - 2b_{-n-2} \sum_{k=0}^{m-1} \hat{T}_{-k} \\ &= -b_{-n} \sum_{k=0}^{m-2} \hat{T}_{-k} - (-4b_{-n-1} + 2b_{-n-2}) \sum_{k=0}^{m-1} \hat{T}_{-k} - 2b_{-n-1} \sum_{k=0}^m \hat{T}_{-k}. \end{aligned}$$

Proof. From the equation $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$, we see that an element of C_{-n-m} is the product of row C_{-n} and a column B_{-m} . From the last equation, we say that an element of C_{-n-m} is the product of a row C_{-n} and column B_{-m} . We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{-n-m} and $C_{-n}B_{-m}$. This completes the proof. □

Corollary 7.8. For $m, n \geq 0$, we have

$$\begin{aligned} \widehat{T}_{-n-m} &= -\widehat{T}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{T}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{T}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \\ \widehat{K}_{-n-m} &= -\widehat{K}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{K}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{K}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \\ \widehat{M}_{-n-m} &= -\widehat{M}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{M}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{M}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \\ \widehat{U}_{-n-m} &= -\widehat{U}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{U}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{U}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \\ \widehat{G}_{-n-m} &= -\widehat{G}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{G}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{G}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \\ \widehat{H}_{-n-m} &= -\widehat{H}_{-n} \sum_{k=0}^{m-2} \widehat{T}_{-k} - \widehat{H}_{-n-1} \left(-4 \sum_{k=0}^{m-1} \widehat{T}_{-k} + 2 \sum_{k=0}^m \widehat{T}_{-k} \right) - 2\widehat{H}_{-n-2} \sum_{k=0}^{m-1} \widehat{T}_{-k}, \end{aligned}$$

From Corollary 6.7, we know that for $n \geq 1$,

$$\sum_{k=1}^n \widehat{T}_{-k} = -2\widehat{T}_{-n-1} + 2\widehat{T}_{-n-2} - 2\widehat{T}_{-n-3}.$$

Since $\widehat{T}_0 = 0$, it follows that

$$\sum_{k=0}^n \widehat{T}_{-k} = -2\widehat{T}_{-n-1} + 2\widehat{T}_{-n-2} - 2\widehat{T}_{-n-3}.$$

So, Theorem 7.7 and Corollary 7.8 can be written in the following forms.

Theorem 7.9. For $m, n \geq 0$, we have

$$\begin{aligned} b_{-n-m} &= b_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) \\ &\quad + b_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) + b_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}). \end{aligned} \tag{7.7}$$

Remark 7.2. By induction, it can be proved that for all integers $m, n \leq 0$, (7.7) holds. So, for all integers m, n , (7.7) is true.

Corollary 7.10. For all integers m, n , we have

$$\begin{aligned} \widehat{T}_{-n-m} &= \widehat{T}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{T}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{T}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}), \\ \widehat{K}_{-n-m} &= \widehat{K}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{K}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{K}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}), \\ \widehat{M}_{-n-m} &= \widehat{M}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{M}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{M}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}), \\ \widehat{U}_{-n-m} &= \widehat{U}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{U}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{U}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}), \\ \widehat{G}_{-n-m} &= \widehat{G}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{G}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{G}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}), \\ \widehat{H}_{-n-m} &= \widehat{H}_{-n}(2\widehat{T}_{-m+1} - 2\widehat{T}_{-m} + 2\widehat{T}_{-m-1}) + \widehat{H}_{-n-1}(-8\widehat{T}_{-m} + 12\widehat{T}_{-m-1} - 12\widehat{T}_{-m-2} + 4\widehat{T}_{-m-3}) \\ &\quad + \widehat{H}_{-n-2}(4\widehat{T}_{-m} - 4\widehat{T}_{-m-1} + 4\widehat{T}_{-m-2}). \end{aligned}$$

8 Conclusions

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduced the binomial transform of the generalized Tribonacci sequence and as special cases, the binomial transform of the Tribonacci, Tribonacci-Lucas, Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas sequences has been defined.

- In section 1, we present some background about the generalized 3-step Fibonacci numbers (also called the generalized Tribonacci numbers).
- In section 2, we defined the binomial transform of the generalized Tribonacci sequence.
- In section 3, we gave Binet's formulas and generating functions of the binomial transform of the generalized Tribonacci sequence.
- In section 4, we present Simson formulas of the binomial transform of the generalized Tribonacci sequence.
- In section 5, we obtained some identities of the binomial transform of the generalized Tribonacci sequence.
- In section 6, we present sum formulas of the binomial transform of the generalized Tribonacci sequence.
- In section 7, we gave some matrix formulation of the binomial transform of the generalized Tribonacci sequence.

Competing Interests

Author has declared that no competing interests exist.

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