# Fuzzy Alexandrov Topologies Associated to Fuzzy Interval Orders

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*Authors' contributions*

*This work was carried out in collaboration among all authors. Author GB proved the results. Author CP and Author MEZ wrote the introduction, provided the final version of the manuscript, and managed the literature searches. All authors read and approved the final manuscript.*

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## Abstract

We characterize the fuzzy $T_0$ - *Alexandrov topologies* on a crisp set $X$, which are associated to *fuzzy interval orders* $R$ on $X$. In this way, we generalize a well known result by Rabinovitch (1978), according to which a crisp partial order is a crisp interval order if and only if the family of all the strict upper sections of the partial order is nested.

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# 1  Introduction

Separation axioms in fuzzy topological spaces were first introduced by [1], and then studied by [2]. In this paper, we are based on the definition of *fuzzy Alexandrov topological space*, which may be viewed as a generalization of the corresponding property concerning topological spaces (see [3]). According to such property, a topology $\tau$ on a set $X$ is said to be *Alexandrov* if it is closed under arbitrary intersections.

We show that there is a one-to-one correspondence between fuzzy interval orders on a nonempty set $X$ and fuzzy $T_0$ - *Alexandrov* topologies on $X$, with minimal bases $\{U_x\}_{x \in X}$ such that the corresponding family $\{U_x^*\}_{x \in X}$ of *partial fuzzy subsets of $X$* is nested. Here, for every $x \in X$, $U_x^*$ is the restriction of $U_x$ to $X \setminus \{x\}$. In our characterization, the concept of a *partial fuzzy set* is crucial, in order to exactly recover the notion of *strict upper sections* of partial orders in the crisp case.

In this way, we generalize a well known result by [4], according to which a crisp partial order is a crisp interval order if and only if the family of all the strict upper sections of the partial order is nested (i.e., linearly ordered by set inclusion).

We recall that the concept of *Alexandrov* topology has been recently considered by [5] in connection to *fuzzy approximation spaces* (i.e., *fuzzy related spaces*) and *rough sets*, as well as by [6], who considered suitable topologies in order to recover *upper semicontinuous Richter-Peleg multi-utility representation* of nontotal preorders. *Fuzzy approximating spaces* (i.e., fuzzy related spaces) have been studied by [7].

# 2  Notation and Preliminaries

A *fuzzy topology* on a nonempty crisp set $X$ is a collection of *fuzzy subsets* of $X$ (i.e. functions from $X$ into the unit interval $[0, 1]$), which is closed under finite intersections (infima) and arbitrary unions (suprema) and which contains both $X$ (i.e., the constant equal to one) and $\emptyset$ (i.e., the constant equal to zero).

For later use, we recall that for two fuzzy subsets $A, B : X \to [0, 1]$ of $X$, $A \subset B$ if and only if $A(x) \le B(x)$ for all $x \in X$.

Let us now introduce the definitions of a *partial fuzzy subset* of $X$, and *inclusion* of a partial fuzzy set into another. These concepts will be used in the fundamental Theorem 3.1 below.

**Definition 2.1.** A *partial fuzzy subset of $X$* is a *partial mapping* $A : (X \supset) X' \to [0, 1]$, i.e., $A$ is restricted to a set $X' \subset X$. In this case, if $x \in X \setminus X'$, then we shall write $A(x) = \emptyset$.

For two partial fuzzy sets $A, B : X \to [0, 1]$,

$$A \subset B \Leftrightarrow A(x) \le B(x) \text{ for every } x \in X \text{ such that } A(x) \ne \emptyset \text{ and } B(x) \ne \emptyset.$$

Therefore, when $A, B$ are partial fuzzy subsets of $X$, $A$ is contained in $B$ whenever $A(x) \le B(x)$ for every point $x$ belonging to both the domains of $A$ and $B$.

**Definition 2.2** ([8])**.** A family $\{A_i\}_{i \in I}$ of (partial) fuzzy subsets of $X$ is said to be *nested* (or equivalently, *linearly ordered by set inclusion*) if, for all $i, j \in I$, either $A_1 \subset A_j$ or $A_j \subset A_i$.

**Definition 2.3** ([9])**.** Let $X$ be a nonempty set. A *fuzzy point* $x_\alpha$ ($x \in X$ and $\alpha \in ]0, 1[$) is a fuzzy subset of $X$ such that $x_\alpha(x) = \alpha$, and $x_\alpha(z) = 0$ for every $z \in X$, $z \ne x$. $x$ and $\alpha$ are respectively called the *support* and *value* of the fuzzy point $x_\alpha$. A fuzzy point $x_\alpha$ is said to *belong* to a fuzzy subset $A$ of $X$ ($x_\alpha \in A$) if $\alpha < A(x)$.

**Definition 2.4** ([9])**.** Let $(X, \tau)$ be a fuzzy topological space. A subfamily $\mathcal{B}_{x_\alpha}$ of $\tau$ is a *local base* of $x_\alpha$ if $x_\alpha \in B$ for every $B \in \mathcal{B}_{x_\alpha}$ and if, whenever $x_\alpha \in U$, $U \in \tau$, then there exists $B \in \mathcal{B}_{x_\alpha}$ such that $x_\alpha \in B \subseteq U$. A subfamily $\mathcal{B}$ of $\tau$ is a *base* for $\tau$ if, for every $U \in \tau$ and for every $x_\alpha \in U$, there exists $B \in \mathcal{B}$ such that $x_\alpha \in B \subseteq U$.

**Definition 2.5** ([2])**.** A fuzzy topological space $(X, \tau)$ is said to be *fuzzy $T_0$* if, whenever $x, y \in X$, $x \neq y$, then there exists $U \in \tau$ such that either $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1$.

Now let us introduce the definition of *fuzzy Alexandrov* topology, which is analogous to the corresponding notion concerning topological spaces.

**Definition 2.6.** [[5]] A fuzzy topological space $(X, \tau)$ is said to be *fuzzy Alexandrov* if $\tau$ is closed under arbitrary intersections (infima).

We recall that the concept of *Alexandrov topological space* was introduced and studied by [3].

**Proposition 2.1.** *If $(X, \tau)$ is a fuzzy Alexandrov topological space, then there exists a unique minimal base $\mathcal{B}$ of $\tau$.*

*Proof.* Define, for every $x \in X$ and for every $\alpha \in ]0, 1[$,

$$B_x^\alpha = \inf_{\{A_i \in \tau : x_\alpha \in A_i\}} A_i.$$

It is easily seen that $\mathcal{B} = \{B_x^\alpha : x \in X, \alpha \in ]0, 1[\}$ is a base of $\tau$. If $x_\alpha \in U$, $U \in \tau$, then consider any real number $\alpha'$ such that $\alpha < \alpha' < U(x)$. Hence $x_\alpha \in B_x^{\alpha'} \subseteq U$. It is straightforward to prove that $\mathcal{B}$ is the unique minimal base of $\tau$. So the proof is complete. $\square$

# 3 Fuzzy Partial Orders and Fuzzy Interval Orders

The fundamental concept of a fuzzy binary relation was introduced and studied by Zadeh (see e.g. [10]). In this section we first recall the definition of a *fuzzy partial order*, and then investigate the relationship between fuzzy partial orders (in particular, interval orders) and fuzzy $T_0$ - Alexandrov topologies.

**Definition 3.1** ([10])**.** Let $R$ be a fuzzy binary relation on a nonempty set $X$ (i.e. $R : X \times X \rightarrow [0, 1]$). $R$ is said to be a *fuzzy partial order* if it is *irreflexive* (i.e. $R(x, x) = 0$ for every $x \in X$) and *min-transitive* (i.e. $R(x, z) \geq min\{R(x, y), R(y, z)\}$ for every $x, y, z \in X$).

*Remark* 3.1. If $R : X \times X \rightarrow [0, 1]$ is an irreflexive binary relation on a crisp set $X$, then, for every pair $(x, y) \in X \times X$, $R(x, y)$ is interpreted as the "degree up to which $x$ is less preferred than $y$".

It is easily seen that a fuzzy partial order is *perfectly antisymmetric*, in the sense that

$$R(x, y) > 0 \Rightarrow R(y, x) = 0 \text{ for every } x, y \in X.$$

We recall that there is a one-to-one correspondence between $T_0$ - Alexandrov topologies on a nonempty set $X$ and crisp partial orders on $X$ (see e.g. [11]). If $R$ is a crisp partial order on $X$ (i.e. $R : X \times X \rightarrow \{0, 1\}$), then define, for every $x \in X$,

$$U_x = \{x\} \cup \{z \in X : R(x, z) = 1\}.$$

We can associate to $R$ the $T_0$ - Alexandrov topology $\tau_R$ with minimal base $(U_x)_{x \in X}$. For a fixed nonempty set $X$, the function $\phi : R \rightarrow \phi(R) = \tau_R$ is a bijection from the set of all the crisp partial orders on $X$ into the set of all the $T_0$ - Alexandrov topologies on $X$.

**Proposition 3.1.** *There is a one-to-one correspondence between fuzzy partial orders on a nonempty set $X$ and fuzzy $T_0$ - Alexandrov topologies on $X$.*

*Proof.* Let $X$ be a nonempty set, and consider a fuzzy $T_0$ - Alexandrov topology $\tau$ on $X$. For every $x \in X$, let

$$U_x = \inf_{\{A_i \in \tau : A_i(x) > 0\}} A_i.$$

Then consider the fuzzy binary relation $R$ on $X$ defined as follows:

$$R(x,y) = \begin{cases} U_x(y) & if \ x \neq y \\ 0 & if \ x = y. \end{cases}$$

Let us prove that $R$ is a fuzzy partial order on $X$. It is clear that $R$ is irreflexive. To prove that $R$ is min-transitive, consider $x, y, z \in X$, and assume that $R(x,y)$ is positive. Hence $R(x,y) = U_x(y)$, and therefore $U_x \supset U_y$ from the definition above. Since $U_x(z) \geq U_y(z)$, it is clear that $R(x,z) \geq min\{R(x,y), R(y,z)\}$.

Conversely, let $R$ be a fuzzy partial order on $X$, and define, for every $x \in X$, the following fuzzy subset $U_x$ of $X$:

$$U_x(y) = \begin{cases} R(x,y) & if \ x \neq y \\ 1 & if \ x = y. \end{cases} \tag{3.1}$$

Denote by $\tau_R$ the fuzzy Alexandrov topology on $X$ with $(U_x)_{x \in X}$ as a minimal base. Using the fact that $R$ is perfectly antisymmetric, it is easy to check that $(X, \tau_R)$ is a fuzzy $T_0$ topological space. If $x \neq y$, then either $U_x(y) = 0$ or $U_x(y) > 0$. In the first case there is nothing to prove since we have $U_x(x) = 1$. In the second case it is $U_y(x) = 0$ since $R$ is perfectly antisymmetric. Moreover, $U_y(y) = 1$ from the definition above. Finally, if $R_1$ and $R_2$ are two different partial orders on $X$, then the $T_0$ Alexandrov topologies $\tau_{R_1}$ and $\tau_{R_2}$ associated with $R_1$ and $R_2$, respectively, are also different. So the proof is complete. $\square$

In the sequel, we shall denote by $\tau_R$ the fuzzy $T_0$ - Alexandrov topology on a crisp set $X$, with a minimal base $\{U_x\}_{x \in X}$ (as defined in equation (3.1)), characterizing a fuzzy partial order $R$ on $X$.

**Definition 3.2** ([12])**.** A fuzzy binary relation $R$ on a crisp set $X$ is said to be a *fuzzy interval order*, if the following conditions are verified:

(i)  $R$ is irreflexive;

(ii)  for every $x, y, z, w \in X$,

$$\max(R(x,w), R(y,z)) \geq \min(R(x,z), R(y,w)).$$

**Theorem 3.1.** *Let $R$ be a fuzzy partial order on a set $X$. Then the following conditions are equivalent:*

(i)  *$R$ is a fuzzy interval order;*

(ii)  *the fuzzy $T_0$ - Alexandrov topology $\tau_R$ on $X$ characterizing $R$ has a unique minimal base $\{U_x\}_{x \in X}$ such that the family $\{U_x^*\}_{x \in X}$ of partial fuzzy subsets of $X$ is nested, where, for every $x \in X$,*

$$U_x^*(y) = \begin{cases} R(x,y) & if \ x \neq y \\ \emptyset & if \ x = y. \end{cases} \tag{3.2}$$

*Proof.* (i) $\Rightarrow$ (ii). First observe that, from the proof of Proposition 3.1, the family $\{U_x\}_{x \in X}$ defined by equation (3.1) is the unique minimal base of the fuzzy $T_0$ - Alexandrov topology $\tau_R$ on $X$ characterizing $R$. By contraposition, assume that the family $\{U_x^*\}_{x \in X}$ $(x \in X)$ is not nested. This

is equivalent to the assertion that there exist two points $x, y \in X$ such that $(U_x^* \not\subset U_y^*)$ $and((U_y^* \not\subset U_x^*))$. Therefore, there exist $z, w \in X$ such that $z, w \neq x$, $z, w \neq y$,

$$(U_x^*(z) = R(x, z) > R(y, z) = U_y^*(z)) \ \ and \ \ (U_y^*(w) = R(y, w) > R(x, w) = U_x^*(w)).$$

Hence, $\max(R(x, w), R(y, z)) < \min(R(x, z), R(y, w))$, and $R$ is not an interval order.

(ii) $\Rightarrow$ (i). We have to show that, for every $x, y, z, w \in X$,

$$\max(R(x, w), R(y, z)) \geq \min(R(x, z), R(y, w)).$$

Without loss of generality, we can assume that $z, w \neq x$, $z, w \neq y$. Consider that

$$\max(R(x, w), R(y, z)) = \max(U_x^*(w), U_y^*(z)).$$

Since the family $\{U_x^*\}_{x \in X}$ $(x \in X)$ of partial fuzzy subsets of $X$ is nested, two cases may occur:

1. $U_x^* \subset U_y^*$: in this case, we have that $U_y^*(z) \geq U_x^*(z)$;

2. $U_y^* \subset U_x^*$: in this case, we have that $U_x^*(w) \geq U_y^*(w)$.

Therefore, in both cases it happens that

$$\max(R(x, w), R(y, z)) = \max(U_x^*(w), U_y^*(z)) \geq \min(U_x^*(z), U_y^*(w)) = \min(R(x, z), R(y, w)).$$

Hence, $R$ is an interval order. This consideration completes the proof. $\square$

*Remark* 3.2. The previous Theorem 3.1 generalizes to the fuzzy case Theorem 2 in [4], according to which a crisp partial $R = \prec$ on a set $X$ is an interval order (i.e., $\prec$ is irreflexive and, for all $x, y, z, w \in X$, $(x \prec z)$ $and(y \prec w) \Rightarrow (x \prec w)$ $or$ $(y \prec z))$ if and only if the family $\{U_x = \{z \in X : x \prec z\}\}_{x \in X}$ of all the *upper sections* of the partial order is nested.

**Example 3.2.** *If $X$ is a finite set, $X = \{x_1, ..., x_n\}$, and $R$ is a fuzzy binary relation on $X$, then $R$ can be described by a matrix in the following way:*

|       | $x_1$        | $x_2$        | $x_3$        | ...  | $x_n$        |
|-------|--------------|--------------|--------------|------|--------------|
| $x_1$ | $R(x_1, x_1)$ | $R(x_1, x_2)$ | $R(x_1, x_3)$ | ...  | $R(x_1, x_n)$ |
| $x_2$ | $R(x_2, x_1)$ | $R(x_2, x_2)$ | $R(x_2, x_3)$ | ...  | $R(x_2, x_n)$ |
| $x_3$ | $R(x_3, x_1)$ | $R(x_3, x_2)$ | $R(x_3, x_3)$ | ...  | $R(x_3, x_n)$ |
| $\vdots$ | ...       | ...          | ...          | ...  | ...          |
| $x_n$ | $R(x_n, x_1)$ | $R(x_n, x_2)$ | $R(x_n, x_3)$ | ...  | $R(x_n, x_n)$ |

$.$

*Consider the fuzzy binary relation $R$ on a five elements set $X = \{x_1, x_2, ..., x_5\}$ described by the matrix*

|       | $x_1$         | $x_2$         | $x_3$ | $x_4$         | $x_5$         |
|-------|---------------|---------------|-------|---------------|---------------|
| $x_1$ | $0$           | $\frac{1}{3}$ | $0$   | $\frac{1}{2}$ | $\frac{1}{6}$ |
| $x_2$ | $0$           | $0$           | $0$   | $\frac{1}{6}$ | $0$           |
| $x_3$ | $\frac{1}{5}$ | $1$           | $0$   | $\frac{3}{4}$ | $\frac{1}{3}$ |
| $x_4$ | $\frac{1}{5}$ | $1$           | $0$   | $\frac{3}{4}$ | $\frac{2}{3}$ |
| $x_5$ | $0$           | $\frac{1}{7}$ | $0$   | $\frac{1}{4}$ | $0$           |

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*Define $U_i^* = U_{x_i}^*$ for $i \in \{1, ..., 5\}$, where we refer to the general definition of the partial fuzzy subsets $U_x^*$ contained in equation (3.1) of Theorem 3.1. Then we have that $U_2^* \subset U_5^* \subset U_1^* \subset U_3^* \subset U_4^*$. Since the family $\{U_i^*\}_{i \in \{1, ..., 5\}}$ is nested, we have that $R$ is a fuzzy interval order on $X$.*

# 4   Conclusions

In this paper we have presented a simple axiomatization of a fuzzy interval order $R$ on a crisp set $X$ in terms of a property of the characterizing fuzzy $T_0$ - Alexandrov topology, according to which the family of all the partial fuzzy subsets of $X$ representing the strict upper sections of $R$ is nested. Our intention is to apply in the future this sort of arguments to fuzzy partial semiorders and fuzzy semiorders, in order to obtain characterizations extending the corresponding ones concerning the crisp cases, as appearing in [11].

# Competing Interests

Authors have declared that no competing interests exist.

# References

[1]   Hutton B, Reilly I. Separation axioms in fuzzy topological spaces. Fuzzy Sets and Systems. 1980;3:93-104.

[2]   Srivastava R, Lal SN, Srivastava AK. On fuzzy $T_0$ and $R_0$ topological spaces. Journal of Mathematical Analysis and Applications. 1988;136:66-73.

[3]   Alexandrov P. Diskrete Raüme. Mat. Sb. N.S. 1937;2:501-518.

[4]   Rabinovitch I. The dimansion of semiorders. Journal of Combinatorial Theory, Ser. A. 1988;25:50-61.

[5]   Wang CY. Topological characterizations of rough fuzzy sets. Fuzzy Sets and Systems. 2017;312:109-125.

[6]   Bosi G, Estevan A, Raventós-Pujol A. Topologies for semicontinuous Richter-Peleg multi-utilities. Theory and Decision. 2020;88:457-470.

[7]   Qin B. Fuzzy approximating spaces. Journal of Applied Mathematics. 2014;2014:7. Article ID 405802

[8]   Zadeh LA. Fuzzy sets. Information and Control. 1975;8:338-353.

[9]   Wong CK. Fuzzy points and local properties of fuzzy Topology. Journal of Mathematical Analysis and Applications. 1974;46:316-328.

[10]  Zadeh LA. Similarity relations and fuzzy orderings. Inf. Sci. 1971;3:17-200.

[11]  Isler R, Bosi G. Topological characterizations of posets. In: Rios, S. Ed., Decision Theory and Decision Analysis: Trends and Challenges, Kluwer. 1994;139-145.

[12]  Mishra S, Srivastava R. Representabilty of fuzzy biorders and fuzzy weak orders. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 2016;24:917-935.

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