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Existence of Nonoscillation Solutions of Higher-Order Nonlinear Neutral Differential Equations

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Authors' contributions

This work was carried out in collaboration between bothl authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we consider the following higher-order nonlinear neutral differential equations:

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)f_1(x(t-\sigma))-Q(t)f_2(x(t-\delta))]=0, \quad t\geq t_0$$

where $\tau,\sigma,\delta\in R^+$, $c\in R, c\neq\pm1$, and $P(t),Q(t)\in C([t_0,\infty),R^+)$, $f_i(u)\in C(R,R)$, $uf_i(u)>0$. we obtain the results which are some sufficient conditions for existence of nonoscillation solutions, special case of the equation has also been studied.

Keywords: Higher-order; differential equation; nonoscillation solutions; existence.

2010 Mathematics Subject Classification: 34K10, 34K11.

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1 Introduction

In this paper, we shall consider existence of nonoscillation solution of higher-order nonlinear neutral differential equations

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)f_1\left(x(t-\sigma)\right)-Q(t)f_2\left(x(t-\delta)\right)]=0, \quad t\geq t_0 \tag{1.1}$$

where $\tau,\sigma,\delta\in R^+$, $c\in R, c\neq\pm1$, and $P(t),Q(t)\in C([t_0,\infty),R^+)$, $R^+=(0,+\infty)$. $f_i(u)\in C(R,R)$, $uf_i(u)>0$. If $u>0$, then $\exists N_i$, st. $0<N_i\leq f_i(u)\leq u$, $|f_i(u)-f_i(v)|\leq L_i|u-v|, i=1,2,$ Let $\mu=\{\tau,\sigma,\delta\}$. By a solution of equation (1.1), we mean a continuously function $x(t)\in C([t_0-\mu,\infty),R)$ for some $t_1\geq t_0$, such that $x(t)+cx(t-\tau)$ is continuously differentiable on $[t_1,\infty)$ and such that equation (1.1) is satisfied for $t\geq t_1$.

Recently, more and more people are interested in nonoscillatory criteria of differential equations, we refer the reader to $[1-11]$, the differential equation in [1].

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)x(t-\sigma)-Q(t)x(t-\delta)]=0, \quad t\geq t_0$$

studied nonoscillation solution for a family of higher-order linear neutral differential equations with positive and negative coefficients, Our principal goal in this paper is to derive existence of nonoscillation solutions for nonlinear equation (1.1).

2 Existence Theorems

Theorem 1. Assume that $0<c<1$ and

$$\int_{t_0}^{\infty}s^{n-1}P(s)ds<\infty, \quad \int_{t_0}^{\infty}s^{n-1}Q(s)ds<\infty. \tag{2.1}$$

Further, assume that there exists a constant $\alpha>\frac{1}{1-c}$ and a sufficiently large $t_1\geq t_0$ such that

$$P(t)\geq\alpha Q(t), \quad for \quad t\geq t_1 \tag{2.2}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.2), there exists a t_1 sufficiently large such that

$$c+\frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(L_1P(s)+L_2Q(s))ds\leq\theta_1<1, \quad for \quad t\geq t_1 \tag{2.3}$$

where θ_1 is a constant, and

$$0\leq\frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(\alpha MP(s)-L_2Q(s))ds\leq c-1-\alpha M, \quad for \quad t\geq t_1 \tag{2.4}$$

$$0\leq\frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}Q(s))ds\leq\frac{1-c-c\alpha M-cM}{\alpha M}, \quad for \quad t\geq t_1 \tag{2.5}$$

hold, where M is positive constant such that

$$\frac{1-c}{\alpha}\leq M\leq\frac{1-c}{c(1+\alpha)} \tag{2.6}$$

holds.

Let X be the set of all continuous and bounded functions on $[t_0,\infty)$ with the norm $\parallel x\parallel=sup_{t\geq t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows:

$$\Omega=\{x\in X: cM\leq x(t)\leq\alpha M, t\geq t_0\}$$

Define an operator $S : \Omega \to X$ as follows:

$$Sx(t) = \begin{cases} 1 - c - cx(t-\tau) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds & t \geq t_1, \\ Sx(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geq t_1$, using (2.4) and (2.6) we get

$$\begin{aligned} Sx(t) =& 1 - c - cx(t-\tau) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds \\ \leq& 1 - c + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(\alpha MP(s) - L_2Q(s))ds \\ \leq& \alpha M \end{aligned}$$

Furthermore, in view of (2.5) and (2.6) we have

$$\begin{aligned} Sx(t) =& 1 - c - cx(t-\tau) + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds \\ \geq& 1 - c - c\alpha M - \frac{M\alpha}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}Q(s)ds \\ \geq& cM \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$.

Now we shall show that operator S is a contraction operator on Ω.

In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned} |Sx(t) - Sy(t)| \leq& c|x(t-\tau) - y(t-\tau)| + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}P(s)|f_1(x(s-\sigma)) - f_1(y(s-\sigma))|ds \\ &+ \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}Q(s)|f_2(x(s-\delta)) - f_2(y(s-\delta))|ds \\ \leq& [c + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}(L_1P(s) + L_2Q(s))ds] \parallel x - y \parallel \\ \leq& \theta_1 \parallel x - y \parallel \end{aligned}$$

This implies that

$$\parallel Sx - Sy \parallel \leq \theta_1 \parallel x - y \parallel$$

where in view of (2.3), $\theta_1 < 1$, which proves that S is a contraction operator on Ω. Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 1.

Theorem 2. Assume that $1 < c < +\infty$ and that (2.1) holds. Further, assume that there exists a constant $\gamma > \frac{c}{c-1}$ and a sufficiently large $t_1 \geq t_0$ such that

$$P(t) \geq \gamma Q(t), \quad for \ \ t \geq t_1 \tag{2.7}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.7), there exists a t_1, sufficiently large such that

$$\frac{1}{c}[1 + \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(L_1p(s) + L_2Q(s))ds] \leq \theta_2 < 1, \quad for \ \ t \geq t_1 \tag{2.8}$$

where θ_2 is a constant, and

$$0 \leq \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(\gamma M_1P(s) - L_2Q(s))ds \leq 1 - c + c\gamma M_1, \quad for \ \ t \geq t_1 \tag{2.9}$$

$$\frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}Q(s)ds < \frac{c-1}{\gamma M_1} - \frac{1}{\gamma} - 1 \tag{2.10}$$

holds, where M_1 is positive constant such that

$$\frac{c-1}{\gamma c} < M_1 < \frac{c-1}{1+\gamma} \tag{2.11}$$

holds. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the norm $\| x \| = sup_{t\geq t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows

$$\Omega = \left\{x \in X : \frac{M_1}{c} \leq x(t) \leq \gamma M_1, t \geq t_0\right\}$$

Define an operator $S : \Omega \to X$ as follows

$$Sx(t) = \begin{cases} 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma))ds & t \geq t_1, \\ Sx(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geq t_1$, using (2.9) and (2.11) we get

$$\begin{aligned} Sx(t) =& 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds \\ \leq& 1 - \frac{1}{c} + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(\gamma M_1 P(s) - L_2 Q(s))ds \\ \leq& \gamma M_1 \end{aligned}$$

Furthermore, in view of (2.10) and (2.11) we have

$$\begin{aligned} Sx(t) =& 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(P(s)f_1(x(s-\delta)) - Q(s)f_2(x(s-\sigma)))ds \\ \geq& 1 - \frac{1}{c} - \frac{\gamma M_1}{c} - \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}\gamma M_1 Q(s)ds \\ \geq& \frac{M_1}{c} \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$. Now we shall show that operator S is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned} |Sx(t) - Sy(t)| \leq& \frac{1}{c}|x(t+\tau) - y(t+\tau)| + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}p(s)|f_1(x(s-\sigma)) - f_1(y(s-\sigma))|ds \\ &+ \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}Q(s)|f_2(x(s-\delta)) - f_2(y(s-\delta))|ds \\ \leq& \frac{1}{c}[1 + \frac{1}{(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}(L_1 p(s) + L_2 Q(s))ds] \| x - y \| \\ \leq& \theta_2 \| x - y \| \end{aligned}$$

This implies that

$$\| Sx - Sy \| \leq \theta_2 \| x - y \|$$

where in view of (2.8), $\theta_2 < 1$, which proves that S is a contraction operator on Ω. Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 2.

Theorem 3. Assume that $-1 < c < 0$ and that (2.1) holds. Further, assume that there exists a constant $\beta > 1$ and a sufficiently large $t_1 \geq t_0$ such that

$$P(t) \geq \beta Q(t), \quad for \;\; t \geq t_1 \tag{2.12}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.12) , there exists a t_1 sufficiently large such that

$$-c+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(L_1p(s)+L_2Q(s))dsdu \le \theta_3 < 1, \quad for \quad t \ge t_1 \tag{2.13}$$

where θ_3 is a constant, and

$$0 \le \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(\beta M_2P(s)-L_2Q(s))dsdu \le (c+1)(\beta M_2-1), \quad for \quad t \ge t_1 \tag{2.14}$$

hold, and

$$\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)ds < \frac{(1+c)(1-M_2)}{\beta M_2} \tag{2.15}$$

where M_2 is positive constant such that

$$\frac{1}{\beta} < M_2 < 1 \tag{2.16}$$

holds. Let X be the set of all continuous and bounded functions on $[t_0,\infty)$ with the norm $\| x \| = sup_{t\ge t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows

$$\Omega = \{x \in X : M_2 \le x(t) \le \beta M_2, t \ge t_0\}$$

Define an operator $S : \Omega \to X$ as follows

$$Sx(t) = \begin{cases} 1+c-cx(t-\tau)+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(P(s)f_1(x(s-\delta))-Q(s)f_2(x(s-\sigma)))ds & t \ge t_1, \\ Sx(t_1) & t_0 \le t \le t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \ge t_1$, using (2.12) and (2.14) we get

$$\begin{aligned} Sx(t) =& 1+c-cx(t-\tau)+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(P(s)f_1(x(s-\delta))-Q(s)f_2(x(s-\sigma)))ds \\ \le& 1+c-c\beta M_2+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(\beta M_2P(s)-L_2Q(s))ds \\ \le& 1+c-c\beta M_2+(c+1)(\beta M_2-1) \\ =& \beta M_2 \end{aligned}$$

Furthermore, in view of (2.15) we have

$$\begin{aligned} Sx(t) =& 1+c-cx(t-\tau)+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(P(s)f_1(x(s-\delta))-Q(s)f_2(x(s-\sigma)))ds \\ \ge& 1+c-cM_2-\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}\beta M_2Q(s)ds \\ \ge& 1+c-cM_2-(1+c)(1-M_2) \\ =& M_2 \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$. Now we shall show that operator S is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned} |Sx(t)-Sy(t)| \le& -c|x(t-\tau)-y(t-\tau)|+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)|f_1(x(s-\sigma))-f_1(y(s-\sigma))|ds \\ &+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)|f_2(x(s-\delta))-f_2(y(s-\delta))|ds \\ \le& [-c+\frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}(L_1p(s)+L_2Q(s))ds] \| x-y \| \\ \le& \theta_3 \| x-y \| \end{aligned}$$

This implies that

$$\| Sx - Sy \| \leq \theta_3 \| x - y \|$$

where in view of (2.13), $\theta_3 < 1$, which proves that S is a contraction operator on Ω. Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This completes the proof of Theorem 3.

Theorem 4. Assume that $-\infty < c < -1$ and that (2.1) holds. Further, assume that there exists a constant $h > 1$ and a sufficiently large $t_1 \geq t_0$ such that

$$P(t) \geq hQ(t), \quad for \;\; t \geq t_1 \tag{2.17}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof : The proof is similar to Theorem 2, we omitted.

By Theorems 1-4, we have the following result

Corollary 1 . Assume that $c \in R, c \neq \pm 1$ and

$$\int_{t_0}^{\infty} s^{n-1} P(s) ds < \infty.$$

then the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + cx(t-\tau)] + (-1)^{n+1}[P(t)f_1(x(t-\sigma))] = 0, \quad t \geq t_0 \tag{2.18}$$

has a bounded nonoscillatory solution.

3 Conclusion

In this paper, we have introuduced existence of nonoscillatory solutions of differential equations of (1.1), the obtained results are easily applicable. If $c = 1$ or $c = -1$, we can study existence of nonoscillatory solutions of differential equations of (1.1) in the future work.

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Competing Interests

Authors have declared that no competing interests exist.

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