



Solution of Two-Point Linear and Nonlinear Boundary Value Problems with Neumann Boundary Conditions Using a New Modified Adomian Decomposition Method

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Authors' contributions

This work was carried out in collaboration between both authors. PAP formulated the problem and JM performed the analysis and computations. Both authors agreed on the final text of the paper.

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Abstract

In this paper, we present the New Modified Adomian Decomposition Method which is a modification of the Modified Adomian Decomposition Method. The new method incorporates the inverse linear operator theorem into the modified Adomian decomposition method for the calculation of u_0 . Six linear and nonlinear boundary value problems with Neumann conditions are solved in order to test the method. The results show that the method is effective.

Keywords: Neumann boundary conditions; Adomian decomposition method; new modified adomian decomposition method.

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1 Introduction

The Adomian Decomposition Method (ADM) is used to find solutions to a wide range of equations. Among the equations where ADM is employed include differential equations [1], [2], [3], [4], integral equations [5], algebraic equations [6], [7], [8], [9] and integral-differential equations or systems of equations. [10], [11]. The ADM is a decomposition method which requires the splitting of the unknown function $u(x)$ into components which are infinite and are expressed as u_0, u_1, u_2, \dots . For nonlinear terms Adomian polynomials denoted by A_n are calculated and they depend on the nonlinearity. The solution of a given problem is then expressed in the form of the equation [12],

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \quad (1.1)$$

The convergence of the solution series (1.1) has been proved by some authors [13], [14]. For example, [13] discussed the convergence of the Adomian method by using the Cauchy Kowalevskaya theorem.

It is started in section 2 of this paper that the ADM has been modified since its introduction. In this paper, we present a modification of the ADM called the New Modified Adomian Decomposition Method (NMADM) which is applied to solve boundary value problems with Neumann conditions. The modification is basically the use of inverse linear operator theorem in combination with the modified Adomian decomposition method. The mathematical analysis of the new method is presented in section 3 of this paper.

2 The Mathematical Analysis of the Adomian Decomposition Method

Let us consider an Initial Value Problem (IVP) of the form,

$$Lu + Ru + Nu = g. \quad (2.1)$$

Here L is the linear operator to be inverted, N represents the nonlinear term, R is the remaining linear operator and g is the source term. We choose $L = \frac{d}{dx}$ and assume that its inverse $L^{-1} = \int_0^x (\cdot) dx$ exists. Solving for u by applying L^{-1} on both sides of equation (2.1) and considering the initial value we get the following equation,

$$u = \varphi(x) + L^{-1}g - L^{-1}[Ru + Nu]. \quad (2.2)$$

The ADM decomposes the solution in the form of equation (1.1), and the nonlinear term Nu is decomposed into a series

$$Nu = \sum_{n=0}^{\infty} A_n. \quad (2.3)$$

Upon substituting (1.1) and (2.3) in (2.2) it gives the following,

$$\sum_{n=0}^{\infty} u_n(x) = \varphi(x) + L^{-1}g - L^{-1} \left[R \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right]. \quad (2.4)$$

The solution components $u_n(x)$ can be determined by the recursive scheme

$$\begin{aligned} u_0(x) &= \varphi(x) + L^{-1}g, \\ u_{n+1} &= -L^{-1}[Ru_n + A_n], n \geq 0. \end{aligned}$$

Thus the $n - term$ approximation of the solution is given by

$$\phi_n(x) = \sum_{k=0}^{n-1} u_k(x). \tag{2.5}$$

Since its introduction there have been several modifications of the method [15], [16], [17]. Here, we describe one of the modifications to the ADM termed as the Modified Adomian Decomposition Method (MADM) [18], [19], [5], [20]. MADM requires that we put the expression $L^{-1} [\sum_{n=0}^{\infty} a_n x^n] - pL^{-1} [\sum_{n=0}^{\infty} a_n x^n]$ into equation (2.4), where p is an artificial parameter and for all $i \in N \cup \{0\}$, a_i are unknown coefficients [21], [22], [23], [24]. We thus obtain the following

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= \varphi(x) + L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - pL^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] \\ &+ L^{-1} g - L^{-1} \left[R \left(\sum_{n=0}^{\infty} u_n \right) + \sum_{n=0}^{\infty} A_n \right]. \end{aligned} \tag{2.6}$$

From equation (2.6), we can define the following recursive scheme

$$\begin{aligned} u_0 &= \varphi(x) + L^{-1} [\sum_{n=0}^{\infty} a_n x^n], \\ u_1 &= L^{-1} g - pL^{-1} [\sum_{n=0}^{\infty} a_n x^n] - L^{-1} [R(u_0) + A_0], \\ u_{n+1} &= -L^{-1} [R(u_n) + A_n]. \end{aligned}$$

To avoid calculation of A_n , $n = 0, 1, 2, \dots$, we calculate the values of the coefficients a_n , for $n = 0, 1, 2, \dots$, by setting $u_1 = 0$. Immediately we verify that $u_n = 0$ for all $n \geq 1$. Setting $p = 1$ we find the solution of equation (2.1) in the form of:

$$u(x) = \varphi(x) + L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right]. \tag{2.7}$$

Unlike the ADM, MADM requires the calculation of u_0 and u_1 only hence reducing the number of iterations, [23], [24], [17].

Let us consider the two-point nonlinear equation of the form.

$$u''(x) + m(x)u'(x) + n(x)h(u(x)) = g(x), \text{ for } x \in [a, b] \tag{2.8}$$

with Nuemann boundary conditions $u'(a) = \beta_1$, $u'(b) = \beta_2$.

We rewrite equation (2.8) as

$$u''(x) = g(x) - m(x)u'(x) - n(x)h(u(x)). \tag{2.9}$$

Integrating the left hand side of equation (2.9) first from x to b and then integrating the resulting expression from a to x and solve for $u(x)$ we obtain the following

$$u(x) = u'(b)x - u'(b)a + u(a) - L_{xx}^{-1} [g(x) - m(x)u'(x) - n(x)h(u(x))]. \tag{2.10}$$

Using MADM we rewrite equation (2.10) as follows

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= u'(b)x - u'(b)a + u(a) + L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - pL^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] \\ &- L^{-1} [g(x)] + L^{-1} \left[\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right]. \end{aligned} \tag{2.11}$$

From equation (2.11) we can define the following recursive scheme

$$\begin{aligned} u_0 &= u'(b)x - u'(b)a + u(a) + L_{xx}^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right], \\ u_1 &= L_{xx}^{-1} [g(x)] - pL_{xx}^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - L_{xx}^{-1} [(u_0) + A_0], \\ u_{n+1} &= -L_{xx}^{-1} [(u_n) + A_n]. \end{aligned}$$

Now in u_0 , the method allows us only to use one boundary condition $u'(b)$ and not $u'(a)$. Hence we can not proceed with MADM. We therefore, propose an alternative procedure to solve equation (2.8). The proposed procedure incorporates the inverse linear operator theorem [25], in calculating u_0 .

This arrangement of this paper is as follows. In section 3, we present the New Modified Adomian Decomposition Method. Section 4 has examples to show the applicability of the new method and the conclusion is presented in section 5.

3 The New Modified Adomian Decomposition Method

In this section we give the analysis of the New Modified Adomian Decomposition Method. It is a modification of ADM as we have incorporated the inverse linear operator theorem in MADM. It solves linear and nonlinear boundary value problems with Neumann conditions with less complications. This is so because the modified scheme avoids the unnecessary computations especially in the calculation of the Adomian polynomials by involving A_0 only in nonlinear cases.

3.1 The Inverse Linear Operator Theorem [25].

This method is based on the MADM and the inverse linear operator theorem [25] which is presented without prove. The proof is in [25].

Theorem If $u'(a) = \alpha$ and $u'(b) = \beta$ are Neumann boundary conditions of a second-order ordinary differential equation then,

$$L_{xx}^{-1} u''(x) = u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega} \int_0^{\Omega} u(x)dx \quad a \leq x \leq b$$

where,

$$L_{xx}^{-1} [\cdot] = \int_{\Omega}^x dx' \int_a^{x'} [\cdot] dx'' + \frac{1}{\Omega} \int_0^{\Omega} dx' (x' \int_b^{x'} [\cdot] dx'')$$

where Ω is an arbitrary finite constant.

3.2 Theoretical Presentation of the New Modified Adomian Decomposition Method

Let us consider equation (2.8). By evaluating the left hand side of equation (2.8) using the inverse linear operator theorem and solving for $u(x)$ we get the following equation,

$$\begin{aligned} u(x) &= (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^{\Omega} u(x)dx + L_{xx}^{-1} g(x) \\ &\quad - L_{xx}^{-1} [m(x)u'(x) + n(x)h(u(x))]. \end{aligned} \tag{3.1}$$

Using MADM, we rewrite equation (3.1) as follows

$$\begin{aligned}
 u(x) = & (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^\Omega u(x)dx + L_{xx}^{-1} \left[\sum_{n=0}^\infty a_n x^n \right] \\
 & - pL_{xx}^{-1} \left[\sum_{n=0}^\infty a_n x^n \right] + L_{xx}^{-1}g(x) - L_{xx}^{-1} [m(x)u'(x) + n(x)h(u(x))].
 \end{aligned}
 \tag{3.2}$$

Substituting equations (1.1) and (2.3) into equation (3.2) gives the following

$$\begin{aligned}
 \sum_{n=0}^\infty u_n(x) = & (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^\Omega u(x)dx + L_{xx}^{-1} \left[\sum_{n=0}^\infty a_n x^n \right] \\
 & - pL_{xx}^{-1} \left[\sum_{n=0}^\infty a_n x^n \right] + L_{xx}^{-1}g(x) - L_{xx}^{-1} \left[\sum_{n=0}^\infty (m(x)u'_n(x)) + \sum_{n=0}^\infty A_n \right].
 \end{aligned}
 \tag{3.3}$$

From equation (3.3) we deduce the following recursive scheme

$$\begin{aligned}
 u_0 = & (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^\Omega u(x)dx + L_{xx}^{-1} [\sum_{n=0}^\infty a_n x^n], \\
 u_1 = & -pL_{xx}^{-1} [\sum_{n=0}^\infty a_n x^n] + L_{xx}^{-1}g(x) - L_{xx}^{-1} [m(x)u'_0(x) + A_0], \\
 & \vdots \\
 u_{n+1} = & -L_{xx}^{-1} [m(x)u'_n(x) + A_n] \quad n \geq 1.
 \end{aligned}$$

It should be noted that in the evaluation of u_0 , $\Omega \rightarrow 0$. We compute the coefficients a_n , $n \geq 0$, by putting $u_1 = 0$ and setting $p = 1$. This yields the solution of equation (2.8) in the form

$$u(x) = x\beta_1 + L_{xx}^{-1} \left[\sum_{n=0}^\infty a_n x^n \right].
 \tag{3.4}$$

4 Illustrations

In this section, seven examples are solved to illustrate the use of NMADM. All the examples are taken from [25], where they are solved using the Advanced Adomian Decomposition Method (AADM). The results confirm the validity of the NMADM.

Example 1

Consider the following linear ordinary boundary problem [25],

$$u''(x) + u(x) + x = 0, \quad 0 \leq x \leq 1
 \tag{4.1}$$

with boundary conditions

$$u'(0) = -1 + \csc(1), \quad u'(1) = -1 + \cot(1).$$

Solution

Let

$$\sum_{n=0}^{\infty} u_n(x) = (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^{\Omega} u(x)dx + L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - pL^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - L_{xx}^{-1}(u(x)) - L_{xx}^{-1}(x). \tag{4.2}$$

Then

$$u_0 = (x - \Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega} \int_0^{\Omega} u(x)dx + L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right]. \tag{4.3}$$

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = -x + x \csc(1) + \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \frac{a_3 x^5}{20} + \frac{a_4 x^6}{30} + \frac{a_5 x^7}{42} + \dots$$

And using equation (4.2), we write u_1 as follows:

$$\begin{aligned} u_1 &= -pL^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - L_{xx}^{-1}(u(x)) - L_{xx}^{-1}(x) \\ &= -\int_0^x \int_0^x (x) dx dx - p \int_0^x \int_0^x \left[\sum_{n=0}^{\infty} a_n x^n \right] dx dx - \int_0^x \int_0^x (u_0) dx dx \\ &= -\frac{1}{6}x^3 - \frac{1}{2}a_0 p x^2 - \frac{1}{6}a_1 p x^3 - \frac{1}{12}a_2 p x^4 - \dots + \frac{1}{6}x^3 - \frac{1}{6}x^3 \csc(1) - \frac{1}{24}a_0 x^4 + \dots \end{aligned}$$

Putting $u_1 = 0$ and $p = 1$, we calculate the values of u_n , for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following,

$$a_0 = 0, \quad a_1 = -\csc(1), \quad a_2 = 0, \quad a_3 = \frac{1}{6} \csc(1), \quad a_4 = 0, \quad a_5 = -\frac{1}{120} \csc(1), \quad \dots$$

The solution to equation (4.1) using equation (3.4) is therefore given by,

$$\begin{aligned} u(x) &= -x + \csc(1) \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \right) \\ &= -x + \csc(1) \sin(x) \end{aligned}$$

as obtained in [25].

Example 2

Consider the following nonlinear BVP, [25].

$$\begin{aligned} u'' - (u')^2 &= 0, \quad 0 \leq x \leq 1 \\ u'(0) &= -1, \quad u'(1) = -\frac{1}{2}. \end{aligned} \tag{4.4}$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = -x + \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \frac{a_3 x^5}{20} + \frac{a_4 x^6}{30} + \frac{a_5 x^7}{42} + \dots$$

Then,

$$\begin{aligned} u_1 &= L^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] - pL^{-1} \left[\sum_{n=0}^{\infty} A_n (u')^2 \right] \\ &= -p \int_0^x \int_0^x \left[\sum_{n=0}^{\infty} a_n x^n \right] - \int_0^x \int_0^x (A_0 (u')^2) \end{aligned}$$

$$= -\frac{1}{2}a_0px^2 - \frac{1}{6}a_1px^3 - \frac{1}{12}a_2px^4 - \dots + \frac{1}{2}x^2 - \frac{1}{3}a_0x^3 - \frac{1}{12}a_1x^4 + \dots .$$

Putting $u_1 = 0$ and $p = 1$, we calculate the values of u_n , for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following,

$$a_0 = 1, \quad a_1 = -2, \quad a_2 = 3, \quad a_3 = -4, \quad \dots .$$

The solution to equation (4.4) using equation (3.4) is found as

$$\begin{aligned} u(x) &= -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 \dots \\ &= -\log(x + 1) \end{aligned}$$

as obtained in [25].

Example 3

Consider the following nonlinear Burger equation, [25].

$$\begin{aligned} u'' + uu' + u &= \frac{1}{2} \sin(2x), \quad 0 \leq x \leq \frac{\pi}{2} \\ u'(0) = 1, \quad u'\left(\frac{\pi}{2}\right) &= 0. \end{aligned} \tag{4.5}$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = x + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + \frac{1}{20}a_3x^5 + \frac{1}{30}a_4x^6 + \frac{1}{42}a_5x^7 + \dots .$$

Then

$$\begin{aligned} u_1 &= -p \left[\sum_{n=0}^{\infty} a_n x^n \right] + L_{xx}^{-1} \left(\frac{\sin(2x)}{2} \right) - L_{xx}^{-1} (A_n(uu')) - L_{xx}^{-1} (u_n). \\ &= -p \int_0^x \int_0^x \left[\sum_{n=0}^{\infty} a_n x^n \right] + \int_0^x \int_0^x \left(\frac{\sin(2x)}{2} \right) - \int_0^x \int_0^x (A_0(uu')) - \int_0^x \int_0^x (u_0), \\ &= -\frac{1}{2}a_0px^2 - \frac{1}{6}a_1px^3 - \frac{1}{12}a_2px^4 - \dots - \frac{1}{4}x + \frac{1}{6}x^3 - \frac{1}{30}x^5 + \dots . \end{aligned}$$

Putting $u_1 = 0$ and $p = 1$, we calculate the values of u_n for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following

$$a_0 = 0, \quad a_1 = -1, \quad a_2 = 0, \quad a_3 = \frac{1}{6}, \quad a_4 = 0, \quad a_5 = -\frac{1}{120}, \quad a_6 = 0, \quad a_7 = \frac{1}{5040}, \quad \dots .$$

Therefore, the solution to equation (4.5) by using equation (3.4) is given by

$$\begin{aligned} u(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots , \\ &= \sin(x), \end{aligned}$$

as obtained in [25].

Example 4

Consider the following linear partial boundary value problems for the heat equation, [25].

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t \geq 0 \\ u_x(0, t) &= e^t, \quad u_x(1, t) = e^t \cosh(1). \end{aligned} \tag{4.6}$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = xe^t + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \frac{a_6x^8}{56} + \frac{a_7x^9}{72} + \dots$$

And

$$\begin{aligned} u_1 &= -pL_{xx}^{-1} [\sum_{n=0}^{\infty} a_n x^n] + L_{xx}^{-1}(u_n). \\ &= -p \int_0^x \int_0^x (a_0 + a_1x + a_2x^2 + \dots) dx dx + \int_0^x \int_0^x \left(xe^t + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \dots\right) dx dx, \\ &= -p \frac{a_0x^2}{2} - p \frac{a_1x^3}{6} - p \frac{a_2x^4}{12} - \dots + \frac{x^3}{6}e^t + \frac{a_0x^4}{24} + \frac{a_1x^5}{120} + \dots \end{aligned}$$

Putting $u_1 = 0$ and $p = 1$, we calculate the values of u_n for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following

$$a_0 = 0, \quad a_1 = e^t, \quad a_2 = 0, \quad a_3 = \frac{e^t}{6}, \quad a_4 = 0, \quad a_5 = \frac{e^t}{120}, \quad a_6 = 0, \quad a_7 = \frac{e^t}{5040}, \quad \dots$$

Therefore, the solution to equation (4.6) using equation (3.4) is given by

$$\begin{aligned} u(x) &= e^t \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362880} + \dots\right), \\ &= e^t \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots\right), \\ &= e^t \sinh(x) \end{aligned}$$

as obtained in [25].

Example 5

Consider the following nonlinear Burger equation [25].

$$u_t + uu_x - u_{xx} = 0, \quad 0 \leq x \leq \frac{\pi}{2}, \quad t \geq 0 \tag{4.7}$$

$$u_x(0, t) = \frac{1}{t} - \frac{\pi^2}{2t^2}, \quad u_x(\frac{\pi}{2}, t) = \frac{1}{t} - \frac{\pi^2}{2t^2} \operatorname{sech}^2\left(\frac{\pi}{t}\right).$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = \frac{x}{t} - \frac{x\pi^2}{2t^2} + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \dots$$

Then.

$$\begin{aligned} u_1 &= -pL_{xx}^{-1} [\sum_{n=0}^{\infty} a_n x^n] + L_{xx}^{-1} \left[\frac{\partial}{\partial t}(u_n) + A_n(uu_x)\right] \\ &= -p \int_0^x \int_0^x (a_0 + a_1x + a_2x^2 + \dots) dx dx + \int_0^x \int_0^x \left(\frac{\partial}{\partial t}u_0 + u_0 \frac{\partial}{\partial x}u_0\right) dx dx \\ &= -p \frac{a_0x^2}{2} - p \frac{a_1x^3}{6} - p \frac{a_2x^4}{12} - \dots - \frac{x^3}{6t^2} + \frac{x^3\pi^2}{6t^3} + \frac{x^3}{6t^2} + \dots \end{aligned}$$

Putting $u_1 = 0$ and $p = 1$, we calculate the values of u_n for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following

$$a_0 = 0, \quad a_1 = \frac{\pi^4}{4t^4}, \quad a_2 = 0, \quad a_3 = \frac{\pi^4}{6t^5} - \frac{\pi^6}{12t^6}, \quad a_4 = 0, \quad a_5 = \frac{\pi^4}{20t^6} - \frac{\pi^2}{20t^7} + \frac{17\pi^8}{960t^8}, \quad \dots$$

Therefore, the solution to equation (4.7) using equation (3.4) is given by

$$u(x) = \frac{x}{t} - \frac{\pi}{t} \left[\left(\frac{\pi x}{2t}\right) - \frac{1}{3} \left(\frac{\pi x}{2t}\right)^3 + \frac{2}{15} \left(\frac{\pi x}{2t}\right)^5 - \frac{17}{315} \left(\frac{\pi x}{2t}\right)^7 + \dots \right] + \xi_7,$$

$$= \frac{x}{t} - \frac{x}{t} \tanh\left(\frac{x\pi}{2t}\right) + \xi_7,$$

where ξ_7 is a constant,

as obtained in [25].

Example 6

Consider the following linear ordinary boundary value problem [25].

$$\begin{aligned} u'' &= e^x \\ u'(0) &= 1 \quad u'(1) = e. \end{aligned} \tag{4.8}$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = x + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \dots$$

Then,

$$\begin{aligned} u_1 &= -pL_{xx}^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] + L_{xx}^{-1} e^x, \\ &= -p \int_0^x \int_0^x (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots) dx dx + \int_0^x \int_0^x e^x dx dx, \\ &= -\frac{a_0px^2}{2} - \frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \frac{a_3px^5}{20} - \frac{a_4px^6}{30} - \frac{a_5px^7}{42} + \dots + e^x - x - 1. \end{aligned}$$

In the expression for u_1 we replace e^x by its Taylor series to get

$$\begin{aligned} u_1 &= -\frac{a_0px^2}{2} - \frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \dots + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right) - x - 1, \\ &= -\frac{a_0px^2}{2} - \frac{a_1px^3}{6} - \frac{a_2px^4}{12} + \dots + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots \end{aligned}$$

Putting $u_1 = 0$ and $p = 1$ we calculate the values of u_n for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}, \quad a_4 = \frac{1}{24}, \quad a_5 = \frac{1}{120}, \dots$$

Thus the solution of equation (4.8) by using equation (3.4) is

$$\begin{aligned} u(x) &= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \dots \\ &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \\ &= e^x \end{aligned}$$

as obtained in [25].

Example 7

Let us consider Bratu equation which is used in some application, [25]. For example it is used in fuel ignition in the thermal combustion theory as well as in the Chandrasekhar model of the expansion of the universe [25].

$$\begin{aligned} u'' - 2e^u &= 0 \quad 0 \leq x \leq 1 \\ u'(0) &= 0, \quad u'(1) = 2 \tan(1). \end{aligned} \tag{4.9}$$

Solution

By using equation (4.3) we calculate u_0 and we obtain the following,

$$u_0 = \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \dots$$

Then

$$\begin{aligned} u_1 &= -pL_{xx}^{-1} \left[\sum_{n=0}^{\infty} a_n x^n \right] + L_{xx}^{-1} e^{u_0} \\ &= -p \int_0^x \int_0^x (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots) dx dx + \int_0^x \int_0^x e^{u_0} dx dx \end{aligned}$$

We replace e^{u_0} by its Taylor series and evaluate to get

$$u_1 = -\frac{a_0 p x^2}{2} - \frac{a_1 p x^3}{6} - \frac{a_2 p x^4}{12} - \dots + x^2 + \frac{1}{12} a_0 x^4 + \frac{1}{360} (3a_0^2 + 2a_2) x^6 + \dots$$

Putting $u_1 = 0$ and $p = 1$ we calculate the values of u_n for $n = 0, 1, 2, 3, 4, 5, \dots$ and we obtain the following

$$a_0 = 2, \quad a_1 = 0, \quad a_2 = 2, \quad a_3 = 0, \quad a_4 = \frac{4}{3}, \quad a_5 = 0, \quad \dots$$

Thus the solution of equation (4.9) by using equation (3.4) is

$$u(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \dots$$

as obtained in [25].

5 Conclusions

We introduced the New Modified Adomian Decomposition Method (NMADM) which is used for solving two-point Boundary Value Problems (BVPs) with Neumann boundary conditions. NMADM is founded on the inverse linear operator theorem and MADM to calculate u_0 and MADM only to find u_1 . To illustrate the applicability and efficiency of the new method, SageMath software for computational work has been used to obtain solutions to several homogeneous and nonhomogeneous differential equations. Noise terms phenomena is applicable to nonhomogeneous differential equations. From the results of the examples illustrated, one can infer that NMADM is an efficient and reliable method for solving BVPs with Neumann boundary conditions.

Competing Interests

Authors have declared that no competing interests exist.

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