Engineering and Applied Science Letters

PSR Press

Article

New fractional Hadamard and Fejér-Hadamard inequalities associated with exponentially (h, m)-convex functions

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Received: 1 March 2020; Accepted: 5 April 2020; Published: 9 April 2020.

Abstract: The aim of this paper is to establish some new fractional Hadamard and Fejér-Hadamard inequalities for exponentially (h, m)-convex functions. These inequalities are produced by using the generalized fractional integral operators containing Mittag-Leffler function via a monotonically increasing function. The presented results hold for various kinds of convexities and well known fractional integral operators.

Keywords: Convex functions, exponentially (h, m)-convex functions, Hadamard inequality, Fejér-Hadamard inequality, generalized fractional integral operators, Mittag-Leffler function.

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1. Introduction and Preliminaries

Convex functions are very important in the field of mathematical inequalities. Nobody can deny the importance of convex functions. A large number of mathematical inequalities exist in literature due to convex functions. For more information related to convex functions and it's properties (see, [1–3]).

Definition 1. A function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be convex, if for all $\alpha, \beta \in I$ and $\kappa \in [0, 1]$, the following inequality holds:

$$\mu(\kappa\alpha + (1-\kappa)\beta) \leq \kappa\mu(\alpha) + (1-\kappa)\mu(\beta). \tag{1}$$

The function μ is said to be concave if $-\mu$ is convex.

A convex function is interpreted very nicely in the coordinate plane by the well known Hadamard inequality stated as follows:

Theorem 2. *Let $\mu : [\alpha, \beta] \to \mathbb{R}$ be a convex function such that $\alpha < \beta$. The following inequalities holds:*

$$\mu\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mu(\kappa)d\kappa \leq \frac{\mu(\alpha)+\mu(\beta)}{2}.$$

In [4], Fejér gave the generalization of Hadamard inequality known as the Fejér-Hadamard inequality stated as follows:

Theorem 3. *Let $\mu : [\alpha, \beta] \to \mathbb{R}$ be a convex function such that $\alpha < \beta$. Also let $\nu : [\alpha, \beta] \to \mathbb{R}$ be a positive, integrable and symmetric to $\frac{\alpha+\beta}{2}$. The following inequalities hold:*

$$\mu\left(\frac{\alpha+\beta}{2}\right)\int_{\alpha}^{\beta}\nu(\kappa)d\kappa \leq \int_{\alpha}^{\beta}\mu(\kappa)\nu(\kappa)d\kappa \leq \frac{\mu(\alpha)+\mu(\beta)}{2}\int_{\alpha}^{\beta}\nu(\kappa)d\kappa. \tag{2}$$

The Hadamard and the Fejér-Hadamard inequalities are further generalized in various ways by using different fractional integral operators such as Riemann-Liouville, Katugampola, conformable and generalized fractional integral operators containing Mittag-Leffler function etc. For more results and details (see, [5–21]).

Next we give the definition of exponentially convex functions.

Definition 4. [9,22] A function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be exponentially convex, if for all $\alpha, \beta \in I$ and $\kappa \in [0,1]$, the following inequality holds:

$$e^{\mu(\kappa\alpha+(1-\kappa)\beta)} \leq \kappa e^{\mu(\alpha)} + (1-\kappa)e^{\mu(\beta)}. \tag{3}$$

In [23], Rashid *et al.*, gave the definition of exponentially s-convex functions.

Definition 5. Let $s \in [0,1]$. A function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be exponentially s-convex, if for all $\alpha, \beta \in I$ and $\kappa \in [0,1]$, the following inequality holds:

$$e^{\mu(\kappa\alpha+(1-\kappa)\beta)} \leq \kappa^s e^{\mu(\alpha)} + (1-\kappa)^s e^{\mu(\beta)}. \tag{4}$$

In [24], Rashid *et al.*, gave the definition of exponentially h-convex functions.

Definition 6. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Then a function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be exponentially h-convex, if for all $\alpha, \beta \in I$ and $\kappa \in [0,1]$, the following inequality holds:

$$e^{\mu(\kappa\alpha+(1-\kappa)\beta)} \leq h(\kappa)e^{\mu(\alpha)} + h(1-\kappa)e^{\mu(\beta)}. \tag{5}$$

In [25], Rashid *et al.*, gave the definition of exponentially m-convex functions.

Definition 7. A function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be exponentially m-convex, if for all $\alpha, \beta \in I$, $m \in (0,1]$ and $\kappa \in [0,1]$, the following inequality holds:

$$e^{\mu(\kappa\alpha+m(1-\kappa)\beta)} \leq \kappa e^{\mu(\alpha)} + m(1-\kappa)e^{\mu(\beta)}. \tag{6}$$

In [26], Rashid *et al.*, gave the definition of exponentially (h,m)-convex functions.

Definition 8. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Then a function $\mu : I \to \mathbb{R}$ on an interval of real line is said to be exponentially (h,m)-convex, if for all $\alpha, \beta \in I$, $m \in (0,1]$ and $\kappa \in [0,1]$, the following inequality holds:

$$e^{\mu(\kappa\alpha+m(1-\kappa)\beta)} \leq h(\kappa)e^{\mu(\alpha)} + mh(1-\kappa)e^{\mu(\beta)}. \tag{7}$$

Remark 1.
1. If we set $h(\kappa) = \kappa$ and $m = 1$ in (7), then exponentially convex function (3) is obtained.
2. If we set $h(\kappa) = \kappa^s$ and $m = 1$ in (7), then exponentially s-convex function (4) is obtained.
3. If we set $m = 1$ in (7), then exponentially h-convex function (5) is obtained.
4. If we set $h(\kappa) = \kappa$ in (7), then exponentially m-convex function (6) is obtained.

Fractional integral operators also play important role in the subject of mathematical analysis. Recently in [27], Andrić *et al.*, defined the generalized fractional integral operators containing generalized Mittag-Leffler function in their kernels as follows:

Definition 9. Let $\psi, \sigma, \phi, l, \varsigma, c \in \mathbb{C}$, $\Re(\sigma), \Re(\phi), \Re(l) > 0$, $\Re(c) > \Re(\varsigma) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\sigma)$. Let $\mu \in L_1[\alpha, \beta]$ and $u \in [\alpha, \beta]$. Then the generalized fractional integral operators $\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\alpha^+}\mu$ and $\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\beta^-}\mu$ are defined by:

$$\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\alpha^+}\mu\right)(u;p) = \int_{\alpha}^{u} (u-\kappa)^{\phi-1} E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(u-\kappa)^{\sigma};p)\mu(\kappa)d\kappa, \tag{8}$$

$$\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\beta^-}\mu\right)(u;p)=\int_u^\beta(\kappa-u)^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(\kappa-u)^\sigma;p)\mu(\kappa)d\kappa, \tag{9}$$

where $E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\kappa;p)$ is the generalized Mittag-Leffler function defined as follows:

$$E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\kappa;p)=\sum_{n=0}^{\infty}\frac{\beta_p(\varsigma+nq,c-\varsigma)}{\beta(\varsigma,c-\varsigma)}\frac{(c)_{nq}}{\Gamma(\sigma n+\phi)}\frac{\kappa^n}{(l)_{nr}}.$$

In [28], Farid defined the following unified integral operators:

Definition 10. Let $\mu,\nu:[\alpha,\beta]\to\mathbb{R}$, $0<\alpha<\beta$ be the functions such that μ be a positive and integrable and ν be a differentiable and strictly increasing. Also, let $\frac{\gamma}{u}$ be an increasing function on $[\alpha,\infty)$ and $\psi,\phi,l,\varsigma,c\in\mathbb{C}$, $\Re(\phi),\Re(l)>0$, $\Re(c)>\Re(\varsigma)>0$ with $p\geq 0$, $\sigma,r>0$ and $0<q\leq r+\sigma$. Then for $u\in[\alpha,\beta]$ the integral operators ${}_\nu\Upsilon^{\gamma,\varsigma,r,q,c}_{\sigma,\phi,l,\alpha^+}\mu$ and ${}_\nu\Upsilon^{\gamma,\varsigma,r,q,c}_{\sigma,\phi,l,\beta^-}\mu$ are defined by:

$$\left({}_\nu\Upsilon^{\gamma,\varsigma,r,q,c}_{\sigma,\phi,l,\alpha^+}\mu\right)(u;p)=\int_\alpha^u\frac{\gamma(\nu(u)-\nu(\kappa))}{\nu(u)-\nu(\kappa)}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(\nu(u)-\nu(\kappa))^\sigma;p)\mu(\kappa)d(\nu(\kappa)), \tag{10}$$

$$\left({}_\nu\Upsilon^{\gamma,\varsigma,r,q,c}_{\sigma,\phi,l,\beta^-}\mu\right)(u;p)=\int_u^\beta\frac{\gamma(\nu(\kappa)-\nu(u))}{\nu(\kappa)-\nu(u)}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(\nu(\kappa)-\nu(u))^\sigma;p)\mu(\kappa)d(\nu(\kappa)). \tag{11}$$

If we set $\gamma(u)=u^\phi$ in (10) and (11), then we get the following generalized fractional integral operators containing Mittag-Leffler function:

Definition 11. Let $\mu,\nu:[\alpha,\beta]\to\mathbb{R}$, $0<\alpha<\beta$ be the functions such that μ be a positive and integrable and ν be a differentiable and strictly increasing. Also let $\psi,\phi,l,\varsigma,c\in\mathbb{C}$, $\Re(\phi),\Re(l)>0$, $\Re(c)>\Re(\varsigma)>0$ with $p\geq 0$, $\sigma,r>0$ and $0<q\leq r+\sigma$. Then for $u\in[\alpha,\beta]$ the integral operators ${}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\alpha^+}\mu$ and ${}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\beta^-}\mu$ are defined by:

$$\left({}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\alpha^+}\mu\right)(u;p)=\int_\alpha^u(\nu(u)-\nu(\kappa))^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(\nu(u)-\nu(\kappa))^\sigma;p)\mu(\kappa)d(\nu(\kappa)), \tag{12}$$

$$\left({}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\beta^-}\mu\right)(u;p)=\int_u^\beta(\nu(\kappa)-\nu(u))^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi(\nu(\kappa)-\nu(u))^\sigma;p)\mu(\kappa)d(\nu(\kappa)). \tag{13}$$

Remark 2. (12) and (13) are the generalization of the following fractional integral operators:

1. Setting $\nu(u)=u$, the fractional integral operators (8) and (9), can be obtained.
2. Setting $\nu(u)=u$ and $p=0$, the fractional integral operators defined by Salim-Faraj in [29], can be obtained.
3. Setting $\nu(u)=u$ and $l=r=1$, the fractional integral operators defined by Rahman *et al.*, in [30], can be obtained.
4. Setting $\nu(u)=u$, $p=0$ and $l=r=1$, the fractional integral operators defined by Srivastava-Tomovski in [31], can be obtained.
5. Setting $\nu(u)=u$, $p=0$ and $l=r=q=1$, the fractional integral operators defined by Prabhakar in [32], can be obtained.
6. Setting $\nu(u)=u$ and $\psi=p=0$, the Riemann-Liouville fractional integral operators can be obtained.

In [33], Mehmood *et al.*, proved the following formulas for constant function:

$$\left({}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\alpha^+}1\right)(u;p)=(\nu(u)-\nu(\alpha))^\phi E^{\varsigma,r,q,c}_{\sigma,\phi+1,l}(\psi(\nu(u)-\nu(\alpha))^\sigma;p):={}_\nu\,\xi^\phi_{\psi,\alpha^+}(u;p), \tag{14}$$

$$\left({}_\nu\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,\beta^-}1\right)(u;p)=(\nu(\beta)-\nu(u))^\phi E^{\varsigma,r,q,c}_{\sigma,\phi+1,l}(\psi(\nu(\beta)-\nu(u))^\sigma;p):={}_\nu\,\xi^\phi_{\psi,\beta^-}(u;p). \tag{15}$$

The objective of this paper is to establish the Hadamard and the Fejér-Hadamard inequalities for generalized fractional integral operators (12) and (13) containing Mittag-Leffler function via a monotone function by using the exponentially (h,m)-convex functions. These inequalities lead to produce the Hadamard and the Fejér-Hadamard inequalities for various kinds of exponentially convexity and well known fractional integral operators given in Remark 1 and Remark 2. In Section 2, we prove the Hadamard inequalities for generalized fractional integral operators (12) and (13) via exponentially (h,m)-convex functions. In Section 3, we prove the Fejér-Hadamard inequalities for these generalized fractional integral operators via exponentially (h,m)-convex functions. Moreover, some of the results published in [26,33,34] have been obtained in particular.

2. Fractional Hadamard inequalities for exponentially (h,m)-convex functions

In this section, we will give two versions of the generalized fractional Hadamard inequality. To establish these inequalities exponentially (h,m)-convexity and generalized fractional integrals operators have been used.

Theorem 12. *Let $\mu,\nu:[\alpha,m\beta]\subset[0,\infty)\to\mathbb{R}$, $0<\alpha<m\beta$ be two functions such that μ be integrable and ν be differentiable. If μ be exponentially (h,m)-convex, ν be strictly increasing and $h\in[0,1]$. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}{}_{\nu}\xi^{\phi}_{\bar{\psi},\alpha^+}(\nu^{-1}(m\nu(\beta));p)\\
&\leq h\left(\frac{1}{2}\right)\left[\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi},\alpha^+}e^{\mu\circ\nu}\right)(\nu^{-1}(m\nu(\beta));p)+m^{\phi+1}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}m^{\sigma},\beta^-}e^{\mu\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\right]\\
&\leq h\left(\frac{1}{2}\right)(m\nu(\beta)-\nu(\alpha))^{\phi}\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,1^-}h\right)(0;p)\right.\\
&\quad\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,0^+}h\right)(1;p)\right],\ \text{where}\ \bar{\psi}=\frac{\psi}{(m\nu(\beta)-\nu(\alpha))^{\sigma}}.
\end{aligned}\tag{16}$$

Proof. By the exponentially (h,m)-convexity of μ, we have

$$e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\leq h\left(\frac{1}{2}\right)\left[e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}+me^{\mu\left((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)\right)}\right].\tag{17}$$

Multiplying (17) with $\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)d\kappa\\
&\leq h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}d\kappa+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)\right)}d\kappa\right].
\end{aligned}\tag{18}$$

Setting $\nu(u)=\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta)$ and $\nu(v)=(1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)$ in (18), then again from exponentially (h,m)-convexity of μ, we have

$$\begin{aligned}
&e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}+me^{\mu\left((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)\right)}\\
&\leq h(\kappa)\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)+mh(1-\kappa)\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right).
\end{aligned}\tag{19}$$

Multiplying (19) with $h\left(\frac{1}{2}\right)\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}d\kappa+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)\right)}d\kappa\right]\\
&\leq h\left(\frac{1}{2}\right)\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h(\kappa)d\kappa+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\right.\\
&\quad\left.\times\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h(1-\kappa)d\kappa\right].
\end{aligned}\tag{20}$$

Setting $\nu(u)=\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta)$ and $\nu(v)=(1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)$ in (20), then by using (8), (9), (12)and (13), the second inequality of (16) is obtained. □

Corollary 1. *Setting $m=1$ in (16), the following inequalities for exponentially h-convex function can be obtained:*

$$e^{\mu\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)}{}_{\nu}\xi^{\phi}_{\bar\psi,\alpha^+}(\beta;p)\le h\left(\frac{1}{2}\right)\left[\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar\psi,\alpha^+}e^{\mu\circ\nu}\right)(\beta;p)+\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar\psi,\beta^-}e^{\mu\circ\nu}\right)(\alpha;p)\right]$$
$$\le h\left(\frac{1}{2}\right)(\nu(\beta)-\nu(\alpha))^{\phi}\left(e^{\mu(\nu(\alpha))}+e^{\mu(\nu(\beta))}\right)\left[\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,1^-}h\right)(0;p)+\left(\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\psi,0^+}h\right)(1;p)\right],\tag{21}$$

where $\bar\psi=\frac{\psi}{(\nu(\beta)-\nu(\alpha))^{\sigma}}$.

Remark 3.
1. If we set $h(\kappa)=\kappa$ in (16), then [33, Theorem 8] is obtained.
2. If we set $h(\kappa)=\kappa$ and $m=1$ in (16), then [33, Corollary 1] is obtained.
3. If we set $\nu(u)=u$ and $h(\kappa)=\kappa$ in (16), then [34, Theorem 2.1] is obtained.
4. If we set $\nu(u)=u$, $h(\kappa)=\kappa$ and $m=1$ in (16), then [34, Corollary 2.2] is obtained.
5. If we set $\nu(u)=u$ in (16), then [26, Theorem 2.1] is obtained.

In the following we give another version of the Hadamard inequality for generalized fractional integral operators via exponentially (h,m)-convex functions.

Theorem 13. *Let $\mu,\nu:[\alpha,m\beta]\subset[0,\infty)\to\mathbb{R}$, $0<\alpha<m\beta$ be two functions such that μ be integrable and ν be differentiable. If μ be exponentially (h,m)-convex and ν be strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}{}_{\nu}\xi^{\phi}_{\bar\psi 2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)\right)^+}(\nu^{-1}(m\nu(\beta));p)$$
$$\le h\left(\frac{1}{2}\right)\left[\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar\psi 2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)\right)^+}e^{\mu\circ\nu}\right)(\nu^{-1}(m\nu(\beta));p)\right.$$
$$\left.+m^{\phi+1}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar\psi(2m)^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+m\nu(\beta)}{2m}\right)\right)^-}e^{\mu\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\right]$$
$$\le h\left(\frac{1}{2}\right)\frac{(m\nu(\beta)-\nu(\alpha))^{\phi}}{2^{\phi}}\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{\kappa}{2}\right)d\kappa\right.$$
$$\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{2-\kappa}{2}\right)d\kappa\right],\tag{22}$$

where $\bar\psi$ is same as in (16).

Proof. By the exponentially (h,m)-convexity of μ, we have

$$e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\le h\left(\frac{1}{2}\right)\left[e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}+me^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}\right].\tag{23}$$

Multiplying (23) with $\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)$ and integrating over $[0,1]$, we have

$$e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)d\kappa$$
$$\le h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}d\kappa+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right].\tag{24}$$

Setting $\nu(u)=\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)$ and $\nu(v)=\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}$ in (24), then by using (12), (13) and (14), the first inequality of (22) is obtained.

Again from exponentially (h,m)-convexity of μ, we have

$$e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}+me^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)} \leq h\left(\frac{\kappa}{2}\right)\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)+mh\left(\frac{2-\kappa}{2}\right)\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right). \tag{25}$$

Multiplying (25) with $h\left(\frac{1}{2}\right)\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}d\kappa+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right]\\
&\leq h\left(\frac{1}{2}\right)\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{\kappa}{2}\right)d\kappa\right.\\
&\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{2-\kappa}{2}\right)d\kappa\right].
\end{aligned} \tag{26}$$

Putting $\nu(u)=\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)$ and $\nu(v)=\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}$ in (26), then by using (12) and (13), the second inequality of (22) is obtained. □

Corollary 2. *Setting $m=1$ in (22), the following inequalities for exponentially h-convex function can be obtained:*

$$\begin{aligned}
&2e^{\mu\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)}{}_{\nu}\xi^{\phi}_{\bar{\psi}2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)\right)^{+}}(\beta;p)\\
&\leq h\left(\frac{1}{2}\right)\left[\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)\right)^{+}}e^{\mu\circ\nu}\right)(\beta;p)+\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)\right)^{-}}e^{\mu\circ\nu}\right)(\alpha;p)\right]\\
&\leq h\left(\frac{1}{2}\right)\frac{(\nu(\beta)-\nu(\alpha))^{\phi}}{2^{\phi}}\left(e^{\mu(\nu(\alpha))}+e^{\mu(\nu(\beta))}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{\kappa}{2}\right)d\kappa+\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)h\left(\frac{2-\kappa}{2}\right)d\kappa\right]
\end{aligned} \tag{27}$$

where $\bar{\psi}$ is same as in (21).

Remark 4.
1. If we set $h(\kappa)=\kappa$ in (22), then [33, Theorem 9] is obtained.
2. If we set $h(\kappa)=\kappa$ and $m=1$ in (22), then [33, Corollary 2] is obtained.
3. If we set $\nu(u)=u$ and $h(\kappa)=\kappa$ in (22), then [34, Theorem 2.4] is obtained.
4. If we set $\nu(u)=u$, $h(\kappa)=\kappa$ and $m=1$ in (22), then [34, Corollary 2.5] is obtained.
5. If we set $\nu(u)=u$ in (22), then [26, Theorem 2.2] is obtained.

3. Fractional Fejér-Hadamard Inequalities for exponentially (h,m)-convex functions

In this section, we will give two versions of the generalized fractional Fejér-Hadamard inequality. To establish these inequalities exponentially (h,m)-convexity and generalized fractional integrals operators have been used.

Theorem 14. *Let $\mu,\nu:[\alpha,m\beta]\subset[0,\infty)\to\mathbb{R}$, $0<\alpha<m\beta$ be two functions such that μ be integrable and ν be differentiable. If μ be exponentially (h,m)-convex and $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$ and ν be strictly increasing. Also, let $\gamma:[\alpha,m\beta]\to\mathbb{R}$ be a function which is non-negative and integrable. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}m^{\sigma},\beta^{-}}e^{\gamma\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\leq h\left(\frac{1}{2}\right)(1+m)\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}m^{\sigma},\beta^{-}}e^{\mu\circ\nu}e^{\gamma\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\\
&\leq h\left(\frac{1}{2}\right)\frac{(m\nu(\beta)-\nu(\alpha))^{\phi}}{m^{\phi}}\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}h(\kappa)\,d\kappa\right.\\
&\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}h(1-\kappa)d\kappa\right],
\end{aligned} \tag{28}$$

where $\bar{\psi}$ is same as in (16).

Proof. Multiplying (17) with $\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}$ and integrating over $[0,1]$, we have

$$e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}d\kappa$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}d\kappa\right.$$
$$\left.+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}d\kappa\right]. \quad (29)$$

Setting $\nu(v)=(1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)$ in (29), then by using (13) and assumption $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$, the first inequality of (28) is obtained.

Now multiplying (19) with $h\left(\frac{1}{2}\right)\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu(\kappa\nu(\alpha)+m(1-\kappa)\nu(\beta))}e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}d\kappa\right.$$
$$\left.+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}d\kappa\right]$$
$$\leq h\left(\frac{1}{2}\right)\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}h(\kappa)\,d\kappa\right.$$
$$\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta))}h(1-\kappa)d\kappa\right]. \quad (30)$$

Setting $\nu(v)=(1-\kappa)\frac{\nu(\alpha)}{m}+\kappa\nu(\beta)$ in (30), then by using (13) and assumption $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$, the second inequality of (28) is obtained. □

Corollary 3. *Setting $m=1$ in (28), the following inequalities for exponentially h-convex function can be obtained:*

$$e^{\mu\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi},\beta^-}e^{\gamma\circ\nu}\right)(\alpha;p)\leq 2h\left(\frac{1}{2}\right)\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi},\beta^-}e^{\mu\circ\nu}e^{\gamma\circ\nu}\right)(\alpha;p)$$
$$\leq h\left(\frac{1}{2}\right)(\nu(\beta)-\nu(\alpha))^{\phi}\left(e^{\mu(\nu(\alpha))}+e^{\mu(\nu(\beta))}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\nu(\alpha)+\kappa\nu(\beta))}h(\kappa)\,d\kappa\right.$$
$$\left.+\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma((1-\kappa)\nu(\alpha)+\kappa\nu(\beta))}h(1-\kappa)\,d\kappa\right], \quad (31)$$

where $\bar{\psi}$ is same as in (21).

Remark 5.
1. If we set $h(\kappa)=\kappa$ in (28), then [33, Theorem 10] is obtained.
2. If we set $h(\kappa)=\kappa$ and $m=1$ in (28), then [33, Corollary 3] is obtained.
3. If we set $\nu(u)=u$ and $h(\kappa)=\kappa$ in (28), then [34, Theorem 2.7] is obtained.
4. If we set $\nu(u)=u$, $h(\kappa)=\kappa$ and $m=1$ in (28), then [34, Corollary 2.8] is obtained.
5. If we set $\nu(u)=u$ in (28), then [26, Theorem 2.3] is obtained.

In the following we give another generalized fractional version of the Fejér-Hadamard inequality.

Theorem 15. *Let $\mu,\nu:[\alpha,m\beta]\subset[0,\infty)\to\mathbb{R}$, $0<\alpha<m\beta$ be two functions such that μ be integrable and ν be differentiable. If μ be exponentially (h,m)-convex and $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$ and ν be strictly increasing. Also, let $\gamma:[\alpha,m\beta]\to\mathbb{R}$ be a function which is non-negative and integrable. Then for generalized fractional integral operators, the following inequalities hold:*

$$
\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}(2m)^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+m\nu(\beta)}{2m}\right)\right)^{-}}e^{\gamma\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\\
&\leq h\left(\frac{1}{2}\right)(1+m)\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}(2m)^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+m\nu(\beta)}{2m}\right)\right)^{-}}e^{\mu\circ\nu}e^{\gamma\circ\nu}\right)\left(\nu^{-1}\left(\frac{\nu(\alpha)}{m}\right);p\right)\\
&\leq h\left(\frac{1}{2}\right)\frac{(m\nu(\beta)-\nu(\alpha))^{\phi}}{(2m)^{\phi}}\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}h\left(\frac{\kappa}{2}\right)d\kappa\right.\\
&\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}h\left(\frac{2-\kappa}{2}\right)d\kappa\right],
\end{aligned}\tag{32}
$$

where $\bar{\psi}$ is same as in (16).

Proof. Multiplying (23) with $\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}$ and integrating over $[0,1]$, we have

$$
\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+m\nu(\beta)}{2}\right)}\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\\
&\leq h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right.\\
&\left.+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right].
\end{aligned}\tag{33}
$$

Setting $\nu(v)=\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}$ in (33), then by using (13) and assumption $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$, the first inequality of (32) is obtained.

Now multiplying (25) with $h\left(\frac{1}{2}\right)\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}$ and integrating over $[0,1]$, we have

$$
\begin{aligned}
&h\left(\frac{1}{2}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\alpha)+m\frac{(2-\kappa)}{2}\nu(\beta)\right)}e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right.\\
&\left.+m\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\mu\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}d\kappa\right]\\
&\leq h\left(\frac{1}{2}\right)\left[\left(e^{\mu(\nu(\alpha))}+me^{\mu(\nu(\beta))}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}h\left(\frac{\kappa}{2}\right)d\kappa\right.\\
&\left.+m\left(e^{\mu(\nu(\beta))}+me^{\mu\left(\frac{\nu(\alpha)}{m^2}\right)}\right)\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}\right)}h\left(\frac{2-\kappa}{2}\right)d\kappa\right].
\end{aligned}\tag{34}
$$

Setting $\nu(v)=\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\frac{\nu(\alpha)}{m}$ in (34), then by using (13) and assumption $\mu(\nu(v))=\mu(\nu(\alpha)+m\nu(\beta)-m\nu(v))$, the second inequality of (32) is obtained. □

Corollary 4. *Setting $m=1$ in (32), the following inequalities for exponentially h-convex function can be obtained:*

$$
\begin{aligned}
&e^{\mu\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)}\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)\right)^{-}}e^{\gamma\circ\nu}\right)(\alpha;p)\leq 2h\left(\frac{1}{2}\right)\left({}_{\nu}\Upsilon^{\varsigma,r,q,c}_{\sigma,\phi,l,\bar{\psi}2^{\sigma},\left(\nu^{-1}\left(\frac{\nu(\alpha)+\nu(\beta)}{2}\right)\right)^{-}}e^{\mu\circ\nu}e^{\gamma\circ\nu}\right)(\alpha;p)\\
&\leq h\left(\frac{1}{2}\right)\frac{(\nu(\beta)-\nu(\alpha))^{\phi}}{2^{\phi}}\left(e^{\mu(\nu(\alpha))}+e^{\mu(\nu(\beta))}\right)\left[\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\nu(\alpha)\right)}h\left(\frac{\kappa}{2}\right)d\kappa\right.\\
&\left.+\int_0^1\kappa^{\phi-1}E^{\varsigma,r,q,c}_{\sigma,\phi,l}(\psi\kappa^{\sigma};p)e^{\gamma\left(\frac{\kappa}{2}\nu(\beta)+\frac{(2-\kappa)}{2}\nu(\alpha)\right)}h\left(\frac{2-\kappa}{2}\right)d\kappa\right],
\end{aligned}\tag{35}
$$

where $\bar{\psi}$ is same as in (21).

Remark 6.
1. If we set $h(\kappa)=\kappa$ in (32), then [33, Theorem 11] is obtained.
2. If we set $h(\kappa)=\kappa$ and $m=1$ in (32), then [33, Corollary 4] is obtained.

Remark 7. By setting $h(\kappa) = \kappa^s$ and $m = 1$ in Theorems 12, 13, 14 and 15, the Hadamard and the Fejér-Hadamard inequalities for exponentially s-convex functions can be obtained. We leave it for interested reader.

4. Concluding remarks

In this article, we established the Hadamard and the Fejér-Hadamard inequalities. To established these inequalities generalized fractional integral operators and exponentially (h, m)-convexity have been used. The presented results hold for various kind of exponentially convexity and well known fractional integral operators given in Remarks 1 and 2. Moreover, the established results have connection with already published results.

Acknowledgments: The research work of the Ghulam Farid is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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