Engineering and Applied Science Letters

PSR Press

Article

On Caputo fractional derivatives via exponential (s, m)-convex functions

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Received: 12 March 2020; Accepted: 21 June 2020; Published: 30 June 2020.

Abstract: In this paper, we establish several integral inequalities including Caputo fractional derivatives for exponential (s, m)-convex functions. By using convexity for exponential (s, m)-convex functions of any positive integer order differentiable function some novel results are obtained.

Keywords: Convex function, exponential (s, m)-convex functions, Caputo-fractional derivatives.

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1. Introduction

Convexity plays an important role in many features of mathematical programming including, for example, sufficient optimality conditions and duality theorems. The topic of convex functions has been treated extensively in the classical book by Hardy, Littlewood and Polya [1]. The study of fractional order derivatives and integrals is called fractional calculus. Fractional calculus have important applications in all fields of applied sciences. Fractional integration and fractional differentiation appear as basic part in the subject of partial differential equations [2,3]. Many types of fractional integral as well as differential operators have been defined in literature. Classical Caputo-fractional derivatives were introduced by Michele Caputo in [4] in 1967. Toader [5] defined the m-convexity as follows:

Definition 1. The function $\Psi : [u, v] \to \mathbb{R}$, is said to be convex, if we have

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau\Psi(z_1) + (1-\tau)\,\Psi(z_2)$$

for all $z_1, z_2 \in [u, v]$ and $\tau \in [0, 1]$.

Definition 2. (see[6]) The function $\Psi : I \subseteq \Re$ is exponential-convex, if

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau e^{-\alpha z_1}\Psi(z_1) + (1-\tau)e^{-\alpha z_2}\Psi(z_2)$$

for all $\tau \in [0, 1]$ and $z_1, z_2 \in I$ and $\alpha \in \Re$.

Definition 3. (see[7]) The function $\Psi : I \subset [0, \infty) \longrightarrow \Re$ is s-convex in second sense with $s \in [0, 1]$, if

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau^s\Psi(z_1) + (1-\tau)^s\Psi(z_2)$$

for all $\tau \in [0, 1)$ and $z_1, z_2 \in I$ and $\alpha \in \Re$.

Definition 4. (see[8]) The function $\Psi : I \subset [0, \infty) \longrightarrow \Re$ is exponential s-convex in second sense with $s \in [0, 1]$, if

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau^s e^{-\beta z_1}\Psi(z_1) + (1-\tau)^s e^{-\beta z_2}\Psi(z_2)$$

for all $\tau \in [0, 1]$ and $z_1, z_2 \in I$ and $\beta \in \Re$.

Definition 5. (see[9]) The function $\Psi : K \to \Re$ is (s,m)-convex in second sense with $s \in [0,1]$, and $K \subseteq [0,\infty]$ be an interval, if

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau^s \Psi(z_1) + (1-\tau)^s m\Psi(z_2)$$

for all $\tau \in [0,1]$ and $z_1, z_2 \in [0,\infty]$.

Definition 6. The function $\Psi : K \to \Re$ is exponential (s,m)-convex in second sense with $s \in [0,1]$, and $K \subseteq [0,\infty]$ be an interval, if

$$\Psi\left(\tau z_1 + (1-\tau)z_2\right) \leq \tau^s e^{-\beta z_1} \Psi(z_1) + (1-\tau)^s e^{-\beta z_2} m\Psi(z_2)$$

for all $\tau \in [0,1]$ and $z_1, z_2 \in [0,\infty]$ and $\beta \in \Re$.

The previous era of fractional calculus is as old as the history of differential calculus. They generalize the differential operators and ordinary integral. However, the fractional derivatives have some basic properties than the corresponding classical ones. On the other hand, besides the smooth requirement, Caputo derivative does not coincide with the classical derivative [10]. We give the following definition of Caputo fractional derivatives, see [2,11–13].

Definition 7. let $\Psi \in AC^n[u,v]$ be a space of functions having nth derivatives absolutely continuous with $\lambda > 0$ and $\lambda \notin \{1,2,3,...\}$, $n = [\lambda] + 1$. The right sided Caputo fractional derivative is as follows:

$$({}^C D^{\lambda}_{u+}\Psi)(z) = \frac{1}{\Gamma(n-\lambda)} \int_u^z \frac{\Psi^{(n)}(\tau)}{(z-\tau)^{\lambda-n+1}} d\tau, z > u. \tag{1}$$

The left sided caputo fractional derivative is as follows:

$$({}^C D^{\lambda}_{v-}\Psi)(z) = \frac{(-1)^n}{\Gamma(n-\lambda)} \int_z^v \frac{\Psi^{(n)}(\tau)}{(\tau-z)^{\lambda-n+1}} d\tau, z < v. \tag{2}$$

The Caputo fractional derivative $({}^C D^{n}_{u+}\Psi)(z)$ coincides with $\Psi^{(n)}(z)$ whereas $({}^C D^{n}_{v-}\Psi)(z)$ coincides with $\Psi^{(n)}(z)$ with exactness to a constant multiplier $(-1)^n$, if $\Lambda = n \in \{1,2,3,...\}$ and usual derivative $\Psi^{(n)}(z)$ of order n exists.

In particular. we have

$$({}^C D^{0}_{u+}\Psi)(z) = ({}^C D^{0}_{v-}\Psi)(z) = \Psi(z) \tag{3}$$

where $n = 1$ and $\lambda = 0$.

In this paper, we establish several new integral inequalities including Caputo fractional derivatives for exponential (s,m)-convex functions. By using convexity for exponential (s,m)-convex functions of any positive integer order differentiable function some novel results are given. The purpose of this paper is to introduce some fractional inequalities for the Caputo-fractional derivatives via (s,m)-convex functions which have derivatives of any integer order.

2. Main Results

First we give the following estimate of the sum of left and right handed Caputo fractional derivatives.

Theorem 1. *Let $f : I \longrightarrow \mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $f^{(n)}$ is a positive (s,m)-convex function, then for $u,v \in I; u < v$ and $\lambda_1, \lambda_2 \geq 1$, the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned} &\Gamma(n-\lambda_1+1)({}^C D^{\lambda_1-1}_{u+} f)(u) + \Gamma(n-\lambda_2+1)({}^C D^{\lambda_2-1}_{v-} f)(u) \\ &\leq \frac{(z-u)^{n-\lambda_1+1}e^{-\beta u} f^{(n)}(u) + (v-z)^{n-\lambda_2+1}e^{-\beta v} f^{(n)}(v)}{s+1} + (m)e^{-\beta z} f^{(n)}(z) \left[\frac{(z-u)^{n-\lambda_1+1} + (v-z)^{n-\lambda_2+1}}{s+1}\right]. \end{aligned} \tag{4}$$

Proof. Let us consider the function f on the interval $[u,z], z \in [u,v]$ and n is a positive integer. For $\tau \in [u,z]$ and $n > \alpha$, the following inequality holds:

$$(z-\tau)^{n-\lambda_1} \le (z-u)^{n-\lambda_1}. \tag{5}$$

Since $f^{(n)}$ is exponential (s,m)-convex therefore for $\tau \in [u,z]$, we have

$$f^{(n)}(\tau) \le \left(\frac{z-\tau}{z-u}\right)^s e^{-\beta u} f^{(n)}(u) + m\left(\frac{\tau-u}{z-u}\right)^s e^{-\beta z} f^{(n)}(z). \tag{6}$$

Multiplying inequalities (6) and (5), then integrating with respect to τ over $[u,z]$, we have

$$\int_u^z (z-\tau)^{n-\lambda_1} f^{(n)}(\tau)d\tau \le \frac{(z-u)^{n-\lambda_1}}{(z-u)^s}\left[e^{-\beta u} f^{(n)}(u)\int_u^z (z-\tau)^s d\tau + me^{-\beta z} f^{(n)}(z)\int_u^z (\tau-u)^s d\tau\right].$$

$$\Gamma(n-\lambda_1+1)({}^C D_{u+}^{\lambda_1-1} f)(z) \le \frac{(z-u)^{n-\lambda_1+1}}{s+1}[e^{-\beta u} f^{(n)}(u) + me^{-\beta z} f^{(n)}(z)]. \tag{7}$$

Now we consider function f on the interval $[z,v], z \in [u,v]$. For $\tau \in [z,v]$, the following inequality holds:

$$(\tau-z)^{n-\lambda_2} \le (v-z)^{n-\lambda_2}. \tag{8}$$

Since $f^{(n)}$ is exponential (s,m)-convex on $[u,v]$, therefore for $\tau \in [z,v]$, we have

$$f^{(n)}(\tau) \le \left(\frac{\tau-z}{v-z}\right)^s e^{-\beta v} f^{(n)}(v) + m\left(\frac{v-\tau}{v-z}\right)^s e^{-\beta z} f^{(n)}(z). \tag{9}$$

Multiplying inequalities (8) and (9), then integrating with respect to τ over $[z,v]$, we have

$$\int_z^v (\tau-z)^{n-\lambda_2} f^{(n)}(\tau)d\tau \le \frac{(v-z)^{n-\lambda_2}}{(v-z)^s}\left[e^{-\beta v} f^{(n)}(v)\int_z^v (\tau-z)^s d\tau + me^{-\beta z} f^{(n)}(z)\int_z^v (v-\tau)^s d\tau\right]$$

$$\Gamma(n-\lambda_2+1)({}^C D_{v-}^{\lambda_2-1} f)(z) \le \frac{(v-z)^{n-\lambda_2+1}}{s+1}[e^{-\beta v} f^{(n)}(v) + me^{-\beta z} f^{(n)}(z)]. \tag{10}$$

Adding (7) and (10) we get the required inequality in (4). □

Corollary 1. *By setting $\lambda_1 = \lambda_2$ in (4) we get the following fractional integral inequality:*

$$\begin{aligned}&\Gamma(n-\lambda_1+1)\left(({}^C D_{u+}^{\lambda_1-1} f)(z) + ({}^C D_{v-}^{\lambda_1-1} f)(z)\right)\\ &\le \frac{(z-u)^{n-\lambda_1+1} e^{-\beta u} f^{(n)}(u) + (v-z)^{n-\lambda_1+1} e^{-\beta v} f^{(n)}(v)}{s+1} + me^{-\beta z} f^{(n)}(z)\left[\frac{(z-u)^{n-\lambda_1+1} + (v-z)^{n-\lambda_1+1}}{s+1}\right].\end{aligned} \tag{11}$$

Remark 1. By setting $s = 1$ the inequality will be of the form:

$$\begin{aligned}&\Gamma(n-\lambda_1+1)\left(({}^C D_{u+}^{\lambda_1-1} f)(z) + ({}^C D_{v-}^{\lambda_1-1} f)(z)\right)\\ &\le \frac{(z-u)^{n-\lambda_1+1} e^{-\beta u} f^{(n)}(u) + (v-z)^{n-\lambda_1+1} e^{-\beta v} f^{(n)}(v)}{2} + me^{-\beta z} f^{(n)}(z)\left[\frac{(z-u)^{n-\lambda_1+1} + (v-z)^{n-\lambda_1+1}}{2}\right].\end{aligned} \tag{12}$$

Remark 2. By setting $\lambda_1 = \lambda_2$, $\beta = 0$, $s = 1$ and $m = 1$, we will get Corollary 2.1 of [14].

Now, we give the next result stated in the following theorem.

Theorem 2. *Let $f : I \longrightarrow \mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $|f^{(n+1)}|$ is exponential (s,m)-convex function, then for $u, v \in I; u < v$ and $\lambda_1, \lambda_2 > 0$, the following inequality for Caputo fractional derivatives holds*

$$\left|\Gamma(n-\lambda_1+1)({}^C D^{\lambda_1}_{u+} f)(z) + \Gamma(n-\lambda_2+1)({}^C D^{\lambda_2}_{v-} f)(z) - \left((z-u)^{n-\lambda_1} f^{(n)}(u) + (v-z)^{n-\lambda_2} f^{(n)}(v)\right)\right|$$
$$\leq \frac{(z-u)^{\lambda_1+1} e^{-\beta u}|f^{(n+1)}(u)| + (v-z)^{\lambda_2+1} e^{-\beta v}|f^{(n+1)}(v)|}{s+1} + m\frac{e^{-\beta z}|f^{(n+1)}(z)|\left((z-u)^{\lambda_1+1} + (v-z)^{\lambda_2+1}\right)}{s+1}. \tag{13}$$

Proof. Since $|f^{(n+1)}|$ is exponential (s, m)-convex function and n is a positive integer, therefore for $\tau \in [u, z]$ and $n > \alpha$, we have

$$|f^{(n+1)}(\tau)| \leq \left(\frac{z-\tau}{z-u}\right)^s e^{-\beta u}|f^{(n+1)}(u)| + m\left(\frac{\tau-u}{z-u}\right)^s e^{-\beta z}|f^{(n+1)}(z)|$$

from which we can write

$$-\left(\left(\frac{z-\tau}{z-u}\right)^s e^{-\beta u}|f^{(n+1)}(u)| + m\left(\frac{\tau-u}{z-u}\right)^s e^{-\beta z}|f^{(n+1)}(z)|\right) \leq f^{(n+1)}(\tau)$$
$$\leq \left(\frac{z-\tau}{z-u}\right)^s e^{-\beta u}|f^{(n+1)}(u)| + m\left(\frac{\tau-u}{z-u}\right)^s e^{-\beta z}|f^{(n+1)}(z)|. \tag{14}$$

We consider the second inequality of inequality (14)

$$f^{(n+1)}(\tau) \leq \left(\frac{z-\tau}{z-u}\right)^s e^{-\beta u}|f^{(n+1)}(u)| + m\left(\frac{\tau-u}{z-u}\right)^s e^{-\beta z}|f^{(n+1)}(z)|. \tag{15}$$

Now for $\lambda_1 > 0$, we have

$$(z-\tau)^{n-\lambda_1} \leq (z-u)^{n-\lambda_1}, \tau \in [u, z]. \tag{16}$$

The product of last two inequalities give

$$(z-\tau)^{n-\lambda_1} f^{(n+1)}(\tau) \leq (z-u)^{n-\lambda_1-s}\left((z-\tau)^s e^{-\beta u}|f^{(n+1)}(u)| + m(\tau-u)^s e^{-\beta z}|f^{(n+1)}(z)|\right).$$

Integrating with respect to τ over $[u, z]$, we have

$$\int_u^z (z-\tau)^{n-\lambda_1} f^{(n+1)}(\tau)d\tau$$
$$\leq (z-u)^{n-\lambda_1-s}\left[e^{-\beta u}|f^{(n+1)}(u)|\int_u^z (z-t\tau)^s d\tau + me^{-\beta z}|f^{(n+1)}(z)|\int_u^z (\tau-u)^s d\tau\right]$$
$$= (z-u)^{n-\lambda_1+1}\left[\frac{e^{-\beta u}|f^{(n+1)}(u)| + me^{-\beta z}|f^{(n+1)}(z)|}{s+1}\right], \tag{17}$$

and

$$\int_u^z (z-\tau)^{n-\lambda_1} f^{(n+1)}(\tau)d\tau = f^{(n)}(\tau)(z-\tau)^{n-\lambda_1}|_u^z + (n-\lambda_1)\int_u^z (z-\tau)^{n-\lambda_1-1} f^{(n)}(\tau)d\tau$$
$$= -f^{(n)}(u)(z-u)^{n-\lambda_1} + \Gamma(n-\lambda_1+1)({}^C D^{\lambda_1}_{u+} f)(z).$$

Therefore (17) takes the form:

$$\Gamma(n-\lambda_1+1)({}^C D^{\lambda_1}_{u+} f)(z) - f^{(n)}(u)(z-u)^{n-\lambda_1} \leq (z-u)^{n-\lambda_1+1}\left[\frac{e^{-\beta u}|f^{(n+1)}(u)| + me^{-\beta z}|f^{(n+1)}(z)|}{s+1}\right]. \tag{18}$$

If one consider from (14) the first inequality and proceed as we did for the second inequality, then following inequality can be obtained:

$$f^{(n)}(u)(z-u)^{n-\lambda_1} - \Gamma(n-\lambda_1+1)({}^C D^{\lambda_1}_{u+} f)(z) \le (z-u)^{n-\lambda_1+1}\left[\frac{e^{-\beta u}|f^{(n+1)}(u)| + me^{-\beta z}|f^{(n+1)}(z)|}{s+1}\right]. \tag{19}$$

From (18) and (19), we get

$$\left|\Gamma(n-\lambda_1+1)({}^C D^{\lambda_1}_{u+} f)(z) - f^{(n)}(u)(z-u)^{n-\lambda_1}\right| \le (z-u)^{n-\lambda_1+1}\left[\frac{e^{-\beta u}|f^{(n+1)}(u)| + me^{-\beta z}|f^{(n+1)}(z)|}{s+1}\right]. \tag{20}$$

On the other hand, for $\tau \in [z,v]$, using convexity of $|f^{(n+1)}|$ as a exponential (s,m)-convex function, we have

$$|f^{(n+1)}(\tau)| \le \left(\frac{\tau-z}{v-z}\right)^s e^{-\beta v}|f^{(n+1)}(v)| + m\left(\frac{v-\tau}{v-z}\right)^s e^{-\beta z}|f^{(n+1)}(z)|. \tag{21}$$

Also for $\tau \in [z,v]$ and $\beta > 0$, we have

$$(\tau-z)^{n-\lambda_2} \le (v-z)^{n-\lambda_2}. \tag{22}$$

By adopting the same treatment as we have done for (14) and (16) one can obtain from (21) and (22) the following inequality:

$$\left|\Gamma(n-\lambda_2+1)({}^C D^{\lambda_2}_{v-} f)(z) - f^{(n)}(v)(v-z)^{n-\lambda_2}\right| \le (v-z)^{n-\lambda_2+1}\left[\frac{e^{-\beta v}|f^{(n+1)}(v)| + me^{-\beta z}|f^{(n+1)}(z)|}{s+1}\right]. \tag{23}$$

By combining the inequalities (20) and (23) via triangular inequality we get the required inequality. □

It is interesting to see the following inequalities as a special case.

Corollary 2. *By setting $\lambda_1 = \lambda_2$ in (13), we get the following fractional integral inequality:*

$$\begin{aligned}
&\left|\Gamma(n-\lambda_1+1)[({}^C D^{\lambda_1}_{u+} f)(z) + ({}^C D^{\lambda_1}_{v-} f)(z)] - \left((z-u)^{n-\lambda_1} f^{(n)}(u) + (v-z)^{n-\lambda_1} f^{(n)}(v)\right)\right| \\
&\le \frac{(z-u)^{n-\lambda_1+1} e^{-\beta u}|f^{(n+1)}(u)| + (v-z)^{n-\lambda_1+1} e^{-\beta v}|f^{(n+1)}(v)|}{s+1} \\
&+ m\frac{e^{-\beta z}|f^{(n+1)}(z)|\left[(z-u)^{n-\lambda_1+1} + (v-z)^{n-\lambda_1+1}\right]}{s+1}.
\end{aligned}$$

Remark 3. By setting $s = 1$ the inequality will be of the form,

$$\begin{aligned}
&\left|\Gamma(n-\lambda_1+1)[({}^C D^{\lambda_1}_{u+} f)(z) + ({}^C D^{\lambda_1}_{v-} f)(z)] - \left((z-u)^{n-\lambda_1} f^{(n)}(u) + (v-z)^{n-\lambda_1} f^{(n)}(v)\right)\right| \\
&\le \frac{(z-u)^{n-\lambda_1+1} e^{-\beta u}|f^{(n+1)}(u)| + (v-z)^{n-\lambda_1+1} e^{-\beta v}|f^{(n+1)}(v)|}{2} \\
&+ m\frac{e^{-\beta z}|f^{(n+1)}(z)|\left[(z-u)^{n-\lambda_1+1} + (v-z)^{n-\lambda_1+1}\right]}{2}.
\end{aligned}$$

Remark 4. By setting $\lambda_1 = \lambda_2$, $\beta = 0$, $s = 1$ and $m = 1$, we will get Corollary 2.2 of [14].

Before going to the next theorem we observe the following result.

Lemma 1. *Let $f : [u,v] \longrightarrow \mathbb{R}$, be a exponential (s,m)-convex function. If f is exponentially symmetric about $\frac{u+v}{2}$, then the following inequality holds*

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{2^s}\left(e^{-\beta z} f(z)\right)(1+m) \quad z \in [u,v]. \tag{24}$$

Proof. Since f is exponential (s,m)-convex we have

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2^s}\left[e^{-\beta(u\tau+(1-\tau)v)}f(u\tau+(1-\tau)v)+me^{-\beta(u(1-\tau)+v\tau)}f(u(1-\tau)+v\tau)\right]. \tag{25}$$

Since f is symmetric about $\frac{a+b}{2}$, therefore we get $f(u+v-z)=f(v\tau+(1-\tau)u)$.

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2^s}\left[e^{-\beta(u\tau+(1-\tau)v)}f(u\tau+(1-\tau)v)+me^{-\beta(u+v-z)}f(u+v-z)\right]. \tag{26}$$

By substituting $z=(u\tau+(1-\tau)v)$ where $z\in[u,v]$, we get

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2^s}\left(e^{-\beta z}f(z)+me^{-\beta(u+v-z)}f(u+v-z)\right).$$

Also f is exponentially symmetric about $\frac{u+v}{2}$, therefore we have $f(u+v-z)=f(z)$ and inequality in (24) holds. □

Theorem 3. *Let $f:I\longrightarrow\mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $f^{(n)}$ is a positive exponential (s,m)- convex and symmetric about $\frac{u+v}{2}$, then for $u,v\in I; u<v$ and $\lambda_1,\lambda_2\geq 1$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\frac{h(\beta)2^s}{2(1+m)}\left(\frac{1}{n-\lambda_1+1}+\frac{1}{n-\lambda_2+1}\right)f^{(n)}\left(\frac{u+v}{2}\right)\\
&\leq \frac{\Gamma(n-\lambda_2+1)({}^C D_{v-}^{\lambda_2-1}f)(u)}{2(v-u)^{n-\lambda_2+1}}+\frac{\Gamma(n-\lambda_1+1)({}^C D_{u+}^{\lambda_1-1}f)(v)}{2(v-u)^{n-\lambda_1+1}}\\
&\leq \frac{mf^{(n)}(u)+f^{(n)}(v)}{(s+1)}.
\end{aligned} \tag{27}$$

where $h(\beta)=e^{\beta v}$ for $\beta<0$ and $h(\beta)=e^{\beta u}$ for $\beta\geq 0$.

Proof. For $z\in[u,v]$, we have

$$(z-a)^{n-\lambda_2}\leq(v-u)^{n-\lambda_2}. \tag{28}$$

Also f is exponential (s,m)-convex function, we have

$$f^{(n)}(z)\leq\left(\frac{z-u}{v-u}\right)^s e^{-\beta v}f^{(n)}(v)+\left(\frac{b-x}{v-u}\right)^s e^{-\beta u}mf^{(n)}(u). \tag{29}$$

Multiplying (28) and (29) and then integrating with respect to z over $[u,v]$, we have

$$\int_u^v (z-u)^{n-\lambda_2}f^{(n)}(z)dz\leq\frac{(v-u)^{n-\lambda_2}}{(v-u)^s}\left(\int_u^v e^{-\beta v}(f^{(n)}(v)(z-u)^s+e^{-\beta u}mf^{(n)}(u)(v-z)^s)dz\right).$$

From which we have

$$\frac{\Gamma(n-\lambda_2+1)({}^C D_{v-}^{\lambda_2-1}f)(u)}{(v-u)^{n-\lambda_2+1}}\leq\frac{e^{-\beta v}f^{(n)}(v)+e^{-\beta u}mf^{(n)}(u)}{s+1}. \tag{30}$$

On the other hand for $z\in[u,v]$ we have

$$(v-z)^{n-\lambda_1}\leq(v-u)^{n-\lambda_1}. \tag{31}$$

Multiplying (29) and (31) and then integrating with respect to z over $[u,v]$, we get

$$\int_u^v (v-z)^{n-\lambda_1}f^{(n)}(z)dz\leq(v-u)^{n-\lambda_1+1}\frac{e^{-\beta u}mf^{(n)}(u)+e^{-\beta v}f^{(n)}(v)}{s+1}.$$

From which we have

$$\frac{\Gamma(n-\lambda_1+1)({}^CD_{u+}^{\lambda_1-1}f)(v)}{(v-u)^{n-\lambda_1+1}} \leq \frac{e^{-\beta u}mf^{(n)}(u)+e^{-\beta v}f^{(n)}(v)}{s+1}. \tag{32}$$

Adding (30) and (32) we get the second inequality.

$$\frac{\Gamma(n-\lambda_2+1)({}^CD_{v-}^{\lambda_2-1}f)(u)}{2(v-u)^{n-\lambda_2+1}}+\frac{\Gamma(n-\lambda_1+1)({}^CD_{u+}^{\lambda_1-1}f)(v)}{2(v-u)^{n-\lambda_1+1}} \leq \frac{e^{-\beta u}mf^{(n)}(u)+e^{-\beta v}f^{(n)}(v)}{s+1}.$$

Since $f^{(n)}$ is exponential s-convex and symmetric about $\frac{u+v}{2}$ using Lemma 1, we have

$$f^{(n)}\left(\frac{u+v}{2}\right) \leq \frac{1}{2^s}\left(e^{-\beta z}f^n(z)(1+m)\right), \quad z\in[u,v]. \tag{33}$$

Multiplying with $(z-u)^{n-\lambda_2}$ on both sides and then integrating over $[u,v]$, we have

$$f^{(n)}\left(\frac{u+v}{2}\right)\int_u^v (z-u)^{n-\lambda_2}dz \leq \frac{(1+m)}{h(\beta)2^s}\int_u^v (z-u)^{n-\lambda_2}f^{(n)}(z)dz. \tag{34}$$

By definition of Caputo fractional derivatives for exponential (s,m)-convex function, one can have

$$f^{(n)}\left(\frac{u+v}{2}\right)\frac{1}{2(n-\lambda_2+1)} \leq \frac{(1+m)}{h(\beta)2^s}\frac{\Gamma(n-\lambda_2+1)({}^CD_{v-}^{\lambda_2-1}f)(u)}{2(v-u)^{n-\lambda_2+1}}. \tag{35}$$

Multiplying (33) with $(v-z)^{n-\lambda_1}$, then integrating over $[u,v]$, one can get

$$f^{(n)}\left(\frac{u+v}{2}\right)\frac{1}{2(n-\lambda_1+1)} \leq \frac{(1+m)}{h(\beta)2^s}\frac{\Gamma(n-\lambda_1+1)({}^CD_{u+}^{\lambda_1-1}f)(v)}{2(v-u)^{n-\lambda_1+1}}. \tag{36}$$

Adding (35) and (36), we get the first inequality. □

Corollary 3. *If we put $\lambda_1=\lambda_2$ in (27), then we get*

$$\frac{h(\beta)2^s}{(1+m)}f^{(n)}\left(\frac{u+v}{2}\right)\frac{1}{(n-\lambda_1+1)} \leq \frac{\Gamma(n-\lambda_1+1)}{(2)(v-u)^{\lambda_1+1}}\left[({}^CD_{v-}^{\lambda_1+1}f)(u)+({}^CD_{u+}^{\lambda_1+1}f)(v)\right] \leq \frac{e^{-\beta u}f^{(n)}(u)+e^{-\beta v}f^{(n)}(v)}{s+1}$$

where $h(\beta)=e^{\beta v}$ for $\beta<0$ and $h(\beta)=e^{\beta u}$ for $\beta\geq 0$.

Remark 5. By setting $\gamma=0$, $s=1$ and $s=1$ in Theorem 3 we will get Theorem 2.3 of [14].

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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