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Characteristic and Moment Generating Functions of Generalised Pareto (GP3) and Weibull Distributions

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Authors' contribution

This work was carried out by author GM. Author CGS supervised the work. Both authors read and approved the final manuscript.

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ABSTRACT

The characteristic functions (*CHFs*) are derived for *GP*3 (generalized *Pareto*) distribution for shape parameters $\xi \neq 0$ and $\xi = 0$ in explicit closed forms. The *CHF* of 3-parameter *Weibull* (type-3extreme value distribution (EVD)) is also derived in a closed form by a direct methodology. Moment-generating functions (*MGFs*) of the distributions are also derived and parametric relations of certain basic properties of the distributions are also obtained. Model estimation by the method of L-moments is also provided.

Keywords: Characteristic function; moment generating function; GP3 distribution; 3-parameter Weibull distribution.

1. INTRODUCTION

Frequencies of extreme event estimations are of particular importance and EVDs play a significant role. The type-3 EVD (3-parameter *Weibull*) is a competing model for the purpose due to its wide applicability. The *GP*3 is also effectively used along with extreme value distributions by many researchers for extreme event estimations [1-9]. In hydrology the *GP*3 is applied to estimate extreme events such as annual maximum rainfall and river discharges

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[10-12]. The threshold estimation in extreme value applications can be well approximated by an extreme value model such as the *GP*3 [13]. A set of algorithms for numerical simulation (synthetic data) of generalised *Pareto* distribution is provided in [14].

But the *CHF*, which is one of the most important properties of a probability distribution, of *GP*3 and its properties are not available in statistics literature. *CHF* has many useful and important properties which give it a central role in statistical theory. It has great theoretical importance and also yields many valuable results in the theory of sampling [15]. It is also known to be the Inverse Fourier Transforms of the probability density function. Thus it provides an alternative route to analytical results instead of dealing directly with probability distributions. In conjunction with the Fast Fourier Transform (FFT), the *CHF* is a first choice for computation of statistical functions [16,17]. This is of significance if only done using the FFT. Goodness-of-fit statistics encountered in the data analysis can be performed with the FFT approximation of a distribution with known *CHF* [17]. The discrete Fourier Transform (DFT) approximation of probability density functions with known *CHFs* is especially useful when analytical expression for the density functions are not available [18]. For numerical approximation of distribution functions, a DFT approximation in terms of *CHF* is applied [18,19]. *CHFs* always exist for all probability density functions unlike the *MGFs*. The *CHF* of the *GP*3 is derived for its shape parameters $\xi \neq 0$ and $\xi = 0$ for the first time. It qualified the tests for a function to be a *CHF* [15,20]. The *MGF* of the *GP*3 is also derived and parametric relations for certain basic properties of a distribution such as raw moments, mean, variance, skewness and kurtosis are also obtained.

CHF of location families suggest that:

$$\varphi_{X-\mu}(t) = exp(it\mu)\varphi_X(t); i = \sqrt{-1}, \mu - location of the probability density function f_{X-\mu}(x) \quad (1)$$

$$\varphi_X(t) - CHF of the probability density function f_X(x) \quad (2)$$

CHF $[\varphi_X(t)]$ of 2 parameter *Weibull* distribution is already derived by Muraleedharan et al. [21,22]. Hence from (1), the *CHF* $[\varphi_{X-\mu}(t)]$ of 3-parameter *Weibull* distribution (type-3 EVD) can be easily obtained.

But here, the *CHF* of type-3 EVD is also derived independently by the simple and lucid methodology adopted for the derivation of *CHF* of *GP*3. The *MGF* of type-3 EVD is also derived and thereby parametric relations are also obtained to estimate the basic properties of the distribution. When location parameter tends to zero, the *CHF* of 3-parameter *Weibull* model tends to the *CHF* of 2-parameter *Weibull* distribution given by Muraleedharan et al. [21,22].

The model estimation by the method of L-moments or linear combination of probability weighted moments (*PWMs*) by Hosking and Wallis [6] is also provided.

The derivations are given in sections 2.1-2.4. The population and sample L-moment estimations are discussed in Sections 3.1 and 3.2. The model estimations of *GP*3 and type-3 EVD are given in sections 3.3 and 3.4. Sections 4 and 5 deal with discussion and conclusion.

2. MATHEMATICAL DERIVATIONS OF CHF AND MGF OF GP3 AND TYPE-3 EVD

2.1 CHF of GP3

The probability density function of generalized *Pareto* distribution (*GP*3) is given as

$$f_X(x)dx = \frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx \text{ for } \xi \neq 0 \tag{3}$$

$$f_X(x)dx = \frac{1}{\sigma}exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] dx \text{ for } \xi = 0 \tag{4}$$

μ isthe locationparameter, σ is the scaleparameter and ξ is the shape parameter respectively

$x \geq \mu$ for $\xi \geq 0$, and $\mu \leq x \leq \mu - \frac{\sigma}{\xi}$ for $\xi < 0$

The cumulative distribution functions of *GP*3 are:

$$F_X(x) = \begin{cases} 1-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} for \xi \neq 0 \\ 1-exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] for \xi = 0 \end{cases} \tag{5}$$

The *CHF* of *GP*3 is initially derived for $\xi \neq 0$ and then for $\xi = 0$.

The *CHF* $\varphi_X(t)$ of *GP*3 ($\xi > 0$) is given as

$$\varphi_X(t) = E[exp(itX)] = \int_\mu^\infty cos(tx) f_X(x)dx + i\int_\mu^\infty sin(tx) f_X(x)\, dx \tag{6}$$

Where X-random variable, *t*-any arbitrary real constant and $i = \sqrt{-1}$

$$\int_\mu^\infty exp(itx) f_X(x)dx = \int_\mu^\infty [cos(tx) + isin(tx)]\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx \tag{7}$$

$$= \int_\mu^\infty cos(tx)\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx + i\int_\mu^\infty sin(tx)\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx$$

Integrating first term of (7) by parts ⇒

$$\int_\mu^\infty cos(tx)\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx = cos(t\mu) - t\int_\mu^\infty sin(tx)\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} dx \tag{8}$$

Now integrating second term of (7) by parts ⇒

$$i\int_\mu^\infty sin(tx)\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} dx = isin(t\mu) + it\int_\mu^\infty cos\,(tx)\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} dx \tag{9}$$

$$(8)+(9) \Rightarrow \int_{\mu}^{\infty} exp(itX) f_X(x)dx = exp(it\mu) + it\left\{\int_{\mu}^{\infty} exp(itx)\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} dx\right\} \qquad (10)$$

After expanding $exp(itx)$andmultiplying with $\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}$ and integrating each product after substituting$\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} = y$ gives the *CHF* of *GP*3 distribution as:

$$\varphi_X(t) = exp(it\mu)\sum_{j=0}^{\infty}\left[\frac{(it\sigma)^j}{\prod_{k=0}^{j}(1-k\xi)}\right], j=0, 1, 2, \ldots \qquad (11)$$

Expression 11 is also the characteristic function of *GP*3 distribution for $\xi < 0$.

The expansion of $\varphi_X(t)$using Taylor's seriesis:

$$\varphi_X(t) = \left(1+\frac{it\mu}{1!}-\frac{t^2\mu^2}{2!}-\cdots\right)\left(1+\frac{it\sigma}{(1-\xi)}-\frac{t^2\sigma^2}{(1-\xi)(1-2\xi)}-\cdots\right) \qquad (12)$$

The *CHF* of *GP*3 distribution for $\xi = 0$ is derived as:

$$\varphi_X(t) = E[exp(itX)] = \int_{\mu}^{\infty} exp(itx)\frac{1}{\sigma}exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]dx \qquad (13)$$

$$= \frac{1}{\sigma}\int_{\mu}^{\infty}[cos(tx)+isin(tx)]exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]dx \qquad (14)$$

$$= exp(it\mu) + it\left\{\int_{\mu}^{\infty} exp(itx)exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]dx\right\} \qquad (15)$$

After expanding $exp(itx)$ and multiplying each term by $exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]$, integrate each product by substituting $exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] = y$. Then the summation of the integrals leads to the characteristic function of *GP*3 distribution for $\xi = 0$ as:

$$\varphi_X(t) = exp(it\mu)\sum_{j=0}^{\infty}(it\sigma)^j \qquad (16)$$

It can also be deduced as the limit, $\xi \rightarrow 0$, of the *CHF* of the *GP*3 distribution for $\xi \neq 0$.

2.2 CHF of Type-3 EVD

Nadarajah and Pogány [23] obtained indirectly the *CHF* of type-3 EVD by a cumbersome procedure that uses the integral referred to as the complex parameter Kratzel function. Muraleedharan [24] also derived the *CHF* of type-3 EVD, but the methodology and the expression are obscure. In this work, the *CHF* of the 3-parameter *Weibull* distribution is derived by the direct and lucid methodology discussed in the previous sections. The probability density function of type-3 EVD is given as:

$$f_X(x)dx = \frac{\xi}{\sigma^{\xi}}(x-\mu)^{\xi-1}exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right]dx; -\infty < \mu < \infty; \sigma, \xi > 0 \qquad (17)$$

Where μ, σ and ξ are location, scale and shape parameters respectively. The cumulative distribution function is given by

$$F_X(x) = 1 - exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right] \quad (18)$$

The *CHF* is derived as

$$\varphi_X(t) = E[exp(itX)] = \int_{\mu}^{\infty} exp(itx) f_X(x)\, dx \quad (19)$$

$$= \int_{\mu}^{\infty} cos(tx)\, f_X(x)dx + i\int_{\mu}^{\infty} sin(tx)\, f_X(x)dx \quad (20)$$

$$= exp(it\mu) + it\int_{\mu}^{\infty} exp(itx) exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right] dx \quad (21)$$

Adding the integrals of the products obtained by multiplying each term of $exp(itx)$ with $exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right]$ and substituting $\left(\frac{x-\mu}{\sigma}\right)^{\xi} = y$ in the integrals of the products leads to the *CHF* of type-3 EVD as:

$$\varphi_X(t) = exp(it\mu) + exp(it\mu)\left[it\sigma\Gamma\left(1+\frac{1}{\xi}\right) + \frac{(it\sigma)^2}{2!}\Gamma\left(1+\frac{2}{\xi}\right) + \frac{(it\sigma)^3}{3!}\Gamma\left(1+\frac{3}{\xi}\right) + \cdots\right] \quad (22)$$

Or

$$\varphi_X(t) = exp(it\mu)\sum_{r=0}^{\infty}\frac{(it\sigma)^r}{r!}\Gamma\left(1+\frac{r}{\xi}\right),\ r = 0,1,2,... \quad (23)$$

When $\mu \to 0$

$$\varphi_X(t) = \sum_{r=0}^{\infty}\frac{(it\sigma)^r}{r!}\Gamma\left(1+\frac{r}{\xi}\right) \quad (24)$$

Ie. If location parameter is zero, then the *CHF* of type-3 EVD (23) tends to the *CHF*of 2-parameter *Weibull* distribution (24) given by Muraleedharan et.al [21, 22].

Revisiting (1):

$\varphi_{X-\mu}(t) = exp(it\mu)\varphi_X(t) \Rightarrow$ thatthe*CHF*oftype-3 EVD follows from 2-parameter *Weibull,* ie. (23) follows from (24), which has been first derived in [21,22].

2.3 MGF of GP3

The *MGF*, $M_X(\theta)$ of *GP*3 distribution (ξ> 0) is derived as

$$M_X(\theta) = E[exp(\theta X)] = \int_{\mu}^{\infty} exp(\theta x)\frac{1}{\sigma}\left[1+\xi\frac{(x-\mu)}{\sigma}\right]^{\frac{-1}{\xi}-1} dx \quad (25)$$

Where θ – arbitrary real constant

Adding the integral of each product obtained by multiplying each term of $exp(\theta x)$ with $\frac{1}{\sigma}\left[1+\xi\frac{(x-\mu)}{\sigma}\right]^{\frac{-1}{\xi}-1}$ and substituting $\left[1+\xi\frac{(x-\mu)}{\sigma}\right]^{\frac{-1}{\xi}}$ =*y* in the integrals of the products leads to the *MGF* of *GP*3 as

$$M_X(\theta) = exp(\theta\mu)\sum_{j=0}^{\infty}\left[\frac{(\theta\sigma)^j}{\prod_{k=0}^{j}(1-k\xi)}\right] \quad (26)$$

Expression (26) is also the *MGF* of *GP*3 for $\xi < 0$.

The raw moments of the *GP*3 distribution can be obtained from $M_X(\theta)$.Ie.

The raw moments of the *GP*3 distribution (ξ>0) can be obtained from $M_X(\theta)$. Ie. The *GP*3 random variable has raw moments up to n^{th} order if $\xi < \frac{1}{n}$.
Then

$$E(X^n) = \mu_n' = M_X^{(n)}(0);\ \xi < \frac{1}{n} \quad (27)$$

μ_n' $-n^{th}$ raw moment

and the *MGF* of *GP*3 (ξ = 0) can also be obtained by the same method as

$$M_X(\theta) = exp(\theta\mu)\sum_{j=0}^{\infty}(\theta\sigma)^j \quad (28)$$

2.4 MGF of Type-3 EVD

Muraleedharan [24] obtained the *MGF* of 3-parameter *Weibull* distribution by deducing from its *CHF* which is also a complex expression. Here the *MGF* of type-3 EVD is derived by a direct methodology as:

$$M_X(\theta) = E[exp(\theta X)] = \int_{\mu}^{\infty} exp(\theta x)\frac{\xi}{\sigma^{\xi}}(x-\mu)^{\xi-1}exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right]dx \quad (29)$$

Adding the integral of each product obtained by multiply each term of $exp(\theta x)$ with $\frac{\xi}{\sigma^{\xi}}(x-\mu)^{\xi-1}exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\xi}\right]$ and substituting $\left(\frac{x-\mu}{\sigma}\right)^{\xi} = y$ in the integrals of the products leads to the *MGF* of Type-3 EVD as

$$M_X(\theta) = exp(\theta\mu)\sum_{r=0}^{\infty}\frac{(\theta\sigma)^r}{r!}\Gamma\left(1+\frac{r}{\xi}\right) \quad (30)$$

When $\mu(location parameter) \to 0$,

$$M_X(\theta) = \sum_{r=0}^{\infty}\frac{(\theta\sigma)^r}{r!}\Gamma\left(1+\frac{r}{\xi}\right) \quad (31)$$

Expression 31 is the *MGF* of 2-parameter *Weibull* distribution.

3. ESTIMATION BY THE METHOD OF L-MOMENTS

3.1 Estimation of L-moments of probability distributions

L-moments or linear combination of probability weighted moments (*PWMs*) are a recent development in statistics and they form the basis of an elegant mathematical theory and facilitate the estimation process. L-moment methods are superior to MLE, method of moments etc. L-moments are more robust to the presence of outliers in the data. L-moments are less subjected to bias in estimation [6].

Analogous to the method of moments, the method of L-moments obtains parameter estimates by equating the first *n* sample L-moments to the population quantities. Hosking et al. [25] and Hosking and Wallis [5] found that with small and moderate samples, the method of L-moments is often more efficient than maximum likelihood (MLE). The method of L-moments yields efficient and computationally convenient estimates of parameters and quantiles.

If *Q* (*p*) is the quantile function of $F_X(x)$, then probability weighted moments α_r are provided by

$$\alpha_r = \int_0^1 Q(p)(1-p)^r dp \qquad (32)$$

Then the L-moments are defined [6] by

$$\lambda_{r+1} = (-1)^r \sum_{k=0}^{r} P_{r,k}^{*} \, \alpha_k \qquad (33)$$

Where

$$P_{r,k}^{*} = \frac{(-1)^{r-k}(r+k)!}{(k!)^2(r-k)!} \qquad (34)$$

Accordingly the first 4 L-moments are given by

$$\lambda_1 = \alpha_0 \qquad (35)$$

$$\lambda_2 = \alpha_0 - 2\alpha_1 \qquad (36)$$

$$\lambda_3 = \alpha_0 - 6\alpha_1 + 6\alpha_2 \qquad (37)$$

$$\lambda_4 = \alpha_0 - 12\alpha_1 + 30\alpha_2 - 20\alpha_3 \qquad (38)$$

The population L-moment measure of location (mean), and L-moment ratio measures of scale (L-CV (τ)), skewness (τ_3) and kurtosis (τ_4) are:

$$\text{Mean} = \lambda_1 \qquad (39)$$

$$\text{Scale} = \tau = \frac{\lambda_2}{\lambda_1} \qquad (40)$$

$$\text{L-skewness} = \tau_3 = \frac{\lambda_3}{\lambda_2} \qquad (41)$$

$$\text{L-kurtosis} = \tau_4 = \frac{\lambda_4}{\lambda_2} \qquad (42)$$

Or in general

$$\tau_r = \frac{\lambda_r}{\lambda_2}, r = 3,4,\ldots \qquad (43)$$

L-moment ratios measure the shape of a distribution independently of its scale of measurement.

3.2 Estimation of Sample L-moments

Let $x_{1,n} \leq x_{2,n} \leq \ldots \leq x_{n,n}$ be the sample in ascending order. Then

$$a_r = n^{-1} \sum_{j=1}^{n} \frac{(n-j)(n-j-1)\ldots(n-j-r+1)}{(n-1)(n-2)\ldots(n-r)} x_{j,n} \tag{44}$$

Ie.

$$a_0 = n^{-1} \sum_{j=1}^{n} x_{j,n} \tag{45}$$

$$a_1 = n^{-1} \sum_{j=1}^{n-1} \frac{(n-j)}{(n-1)} x_{j,n} \tag{46}$$

$$a_2 = n^{-1} \sum_{j=1}^{n-2} \frac{(n-j)(n-j-1)}{(n-1)(n-2)} x_{j,n} \tag{47}$$

and

$$l_{r+1} = (-1)^r \sum_{k=0}^{r} P^*_{r,k} a_k \tag{48}$$

Ie.

$$l_1 = a_0 \tag{49}$$

$$l_2 = a_0 - 2a_1 \tag{50}$$

$$l_3 = a_0 - 6a_1 + 6a_2 \tag{51}$$

$$l_4 = a_0 - 12a_1 + 30a_2 - 20a_3 \tag{52}$$

a_r and l_r are unbiased estimators of α_r and λ_r

The sample L-moment ratios are given by

$$\text{Mean} = l_1 \tag{53}$$

$$\text{L-CV} = t = \frac{l_2}{l_1} \tag{54}$$

$$\text{L-skewness} = t_3 = \frac{l_3}{l_2} \tag{55}$$

$$\text{L-kurtosis} = t_4 = \frac{l_4}{l_2} \tag{56}$$

Or in general

$$t_r = \frac{l_r}{l_2}, r = 3,4, \ldots \tag{57}$$

3.3 Estimation of Parameters of GP3

To estimate the model parameters by the method of L-moments, the corresponding model and sample L-moment ratios are equated and solved for the unknown parameters. Usually the first 3 L-moments ($\lambda_1, \lambda_2, \tau_3$) will be sufficient to estimate a model with 3 parameters. The first 4 L-moments of *GP*3 are given [6] as:

$$\lambda_1 = \mu + \frac{\sigma}{(1-\xi)} \tag{58}$$

$$\lambda_2 = \frac{\sigma}{(1-\xi)(2-\xi)} \tag{59}$$

$$\tau_3 = \frac{(1+\xi)}{(3-\xi)} \tag{60}$$

$$\tau_4 = \frac{(1+\xi)(2+\xi)}{(3-\xi)(4-\xi)} \tag{61}$$

3.4 Estimation of Parameters of Type-3 EVD

The first 4 L-moments of *Weibull* distribution are derived for estimation of the parameters. They are:

$$\lambda_1 = \mu + \sigma\Gamma\left(1 + \frac{1}{\xi}\right) \tag{62}$$

$$\lambda_2 = \sigma\Gamma\left(1 + \frac{1}{\xi}\right)\left(1 - 2^{-\frac{1}{\xi}}\right) \tag{63}$$

$$\tau_3 = \frac{\left(1-3\times2^{-\frac{1}{\xi}}+2\times3^{-\frac{1}{\xi}}\right)}{\left(1-2^{-\frac{1}{\xi}}\right)} \tag{64}$$

$$\tau_4 = \frac{\left(1-6\times2^{-\frac{1}{\xi}}+10\times3^{-\frac{1}{\xi}}-5\times4^{-\frac{1}{\xi}}\right)}{\left(1-2^{-\frac{1}{\xi}}\right)} \tag{65}$$

After equating the corresponding population L-moments with the sample L-moments, the parameters can be estimated numerically.

4. RESULTS AND DISCUSSION

The *GP*3 and type-3 EVD are widely used in extreme event estimations. But the *CHF* of *GP*3 (for shape parameters $\xi \neq 0$ and $\xi = 0$) distribution, which is one of the most important property of a probability distribution, is not available in literature. Hence the *CHFs* of *GP3* are derived in explicit closed forms for the first time. The *CHF* of *GP3* for $\xi < 0$ has the same functional form of the *CHF* of *GP*3 for $\xi > 0$.

The *MGF* of the distributions are also derived to obtain parametric relations for certain basic properties of the distributions such as raw moments, mean, variance, skewness and kurtosis (Tables 1 and 2). The constant skewness and kurtosis of the *GP*3 (ξ=0) are 2.0 and 9.0

respectively. Ie. It has a positive skewness and excess kurtosis 6. Also the skewness and kurtosis are respectively equal to that of the exponential distribution.

The *CHF* of type-3 EVD is also derived by the methodology that is direct and lucid. When μ (location parameter) = 0, the *CHF* of type-3 EVD tends to the *CHF* of 2-parameter *Weibull* distribution. All the 3 *CHFs* derived here qualified the tests for a function to be a *CHF* [15,20]. For example the *CHF* of *GP*3 satisfied the tests such as:

- that $\varphi_X(t)$ must be continuous in *t*

$\varphi_X(t)$ of *GP*3 is clearly continuous in *t*

- that $\varphi_X(t)$ is defined in every finite *t*- interval

$\varphi_X(t)$ of *GP*3 is defined in every finite *t*-interval.

- that $\varphi_X(0) = 1$

$$\varphi_X(t) = exp(it\mu)\sum_{j=0}^{\infty}\left[\frac{(it\sigma)^j}{\prod_{k=0}^{j}(1-k\xi)}\right] = exp(it\mu)\times\left[1+\frac{it\sigma}{(1-\xi)}+\frac{(it\sigma)^2}{(1-\xi)(1-2\xi)}+\cdots\right] \qquad (66)$$

$$\therefore \varphi_X(0) = 1$$

And hence the condition is satisfied.

- that $\varphi_X(t)$ and $\varphi_X(-t)$ shall be conjugate quantities

$$\varphi_X(t) = exp(it\mu)\times\left[1+\frac{it\sigma}{(1-\xi)}+\frac{(it\sigma)^2}{(1-\xi)(1-2\xi)}+\cdots\right] \qquad (67)$$

$$\varphi_X(-t) = exp[-(it\mu)]\times\left[1-\frac{it\sigma}{(1-\xi)}+\frac{(it\sigma)^2}{(1-\xi)(1-2\xi)}-\cdots\right] \qquad (68)$$

$\therefore \varphi_X(t)$and$\varphi_X(-t)$ are conjugate quantities.

- that $|\varphi_X(t)| \leq \int|exp(itX)|dF_X(x) \leq 1 = \varphi_X(0)$

For any random variable *X* with finite mean $\bar{x}$, the *CHF*, by Taylor´s theorem can be written as:

$$\varphi_X(t) = 1 + it\bar{x} + o(t), t \to 0 \qquad (69)$$

Hence *CHF* of *GP*3 can be written as:

$$\varphi_X(t) = 1 + it\left(\mu+\frac{\sigma}{(1-\xi)}\right), t \to 0 \qquad (70)$$

$$\therefore |\varphi_X(t)| = \sqrt{\left(1+t^2\left[\mu+\frac{\sigma}{(1-\xi)}\right]^2\right)} = 1 = \varphi_X(0) \qquad (71)$$

Table 1. The first 4 raw moments of *GP*3, type-3 EVD and 2- parameter *Weibull* distributions

***GP*3 (*ξ*> 0)**	
μ_1'	$\mu+\frac{\sigma}{1-\xi}, \xi<1$
μ_2'	$\mu^2+\frac{2\mu\sigma}{(1-\xi)}+\frac{2\sigma^2}{(1-\xi)(1-2\xi)}\ , \xi<\frac{1}{2}$
μ_3'	$\mu^3+\frac{3\mu^2\sigma}{(1-\xi)}+\frac{6\mu\sigma^2}{(1-\xi)(1-2\xi)}+\frac{6\sigma^3}{(1-\xi)(1-2\xi)(1-3\xi)}, \xi<\frac{1}{3}$
μ_4'	$\mu^4+\frac{4\mu^3\sigma}{(1-\xi)}+\frac{12\mu^2\sigma^2}{(1-\xi)(1-2\xi)}+\frac{24\mu\sigma^3}{(1-\xi)(1-2\xi)(1-3\xi)}+\frac{24\sigma^4}{(1-\xi)(1-2\xi)(1-3\xi)(1-4\xi)}, \xi<\frac{1}{4}$
***GP*3(*ξ*=0)**	
μ_1'	$\mu+\sigma$
μ_2'	$\mu^2+2\mu\sigma+2\sigma^2$
μ_3'	$\mu^3+3\mu^2\sigma+6\mu\sigma^2+6\sigma^3$
μ_4'	$\mu^4+4\mu^3\sigma+12\mu^2\sigma^2+24\mu\sigma^3+24\sigma^4$
Type-3 EVD (3-parameter *Weibull*)	
μ_1'	$\mu+\sigma g_1$
μ_2'	$\mu^2+2\mu\sigma g_1+\sigma^2 g_2$
μ_3'	$\mu^3+3\mu^2\sigma g_1+3\mu\sigma^2 g_2+\sigma^3 g_3$
μ_4'	$\mu^4+4\mu^3\sigma g_1+6\mu^2\sigma^2 g_2+4\mu\sigma^3 g_3+\sigma^4 g_4$
Weibull (2- parameter)	
μ_1'	σg_1
μ_2'	$\sigma^2 g_2$
μ_3'	$\sigma^3 g_3$
μ_4'	$\sigma^4 g_4$

$^*g_k=\Gamma\left(1+\frac{k}{\xi}\right)$

Table 2. Parametric relations of probabilistic models for certain model characteristics

Model characteristics	**Parametric relations of probabilistic models**			
	*GP*3 $\xi>0$	*GP*3 $\xi=0$	Type-3 EVD (3-parameter *Weibull*)	2-parameter *Weibull*
Mean	$\mu+\frac{\sigma}{1-\xi}, \xi<1$	$\mu+\sigma$	$\mu+\sigma g_1$	σg_1
Variance	$\frac{\sigma^2}{(1-\xi)^2(1-2\xi)}, \xi<\frac{1}{2}$	σ^2	$\sigma^2[g_2-g_1^2]$	$\sigma^2[g_2-g_1^2]$
Skewness	$\frac{2(1+\xi)\sqrt{1-2\xi}}{(1-3\xi)}, \xi<\frac{1}{3}$	2	$\frac{g_3-3g_1g_2+2g_1^3}{(g_2-g_1^2)^{\frac{3}{2}}}$	$\frac{g_3-3g_1g_2+2g_1^3}{(g_2-g_1^2)^{\frac{3}{2}}}$
Kurtosis	$\frac{3(1-2\xi)(2\xi^2+\xi+3)}{(1-3\xi)(1-4\xi)}, \xi<\frac{1}{4}$	9	$\frac{g_4-4g_1g_3+6g_2g_1^2-3g_1^4}{(g_2-g_1^2)^2}$	$\frac{g_4-4g_1g_3+6g_2g_1^2-3g_1^4}{(g_2-g_1^2)^2}$

$^*g_k=\Gamma\left(1+\frac{k}{\xi}\right)$

- *CHF* by Taylor´s theorem for $t \to 0$ is also satisfied as:

$$\bar{x} = \mu + \frac{\sigma}{(1-\xi)}, \tag{72}$$

This is the mean of *GP*3 distribution (Table 2).

- If a random variable *X* has moments up to *m*th order, then the *CHF*$\varphi_X(t)$ is *m* times continuously differentiable. Ie.

$$E(X^m) = (-i)^m \varphi_X^m(0) \tag{73}$$

This property is also satisfied by the *CHF* of *GP*3 distribution.

The *MGF* of the type-3 EVD is also derived to obtain the properties of the distribution. The expressions of the first 4 raw moments and the parametric relations of the basic properties of the distributions obtained therein from the *MGF* are given in Tables 1 and 2. By assigning μ = 0, all properties of 2-parameter *Weibull* distribution can also be obtained from the *MGF* of the type-3 EVD.

Model estimations by the method of L-moments are demonstrably superior to the existing previous methods such as MLE, method of moments etc. The first 4 L-moments of *GP*3 and type-3 EVD that facilitates the parameter estimations of the 3 parameter models are provided. The model parameters can be estimated numerically.

5. CONCLUSION

The *GP*3 distribution along with the type-3 EVD is widely applied in extreme event estimations. Hence the *CHFs* of the *GP*3 distribution for shape parameters $\xi \neq 0$ and $\xi = 0$ are derived in explicit closed forms. The *CHF* of type-3 EVD is also derived in a closed form which is simple and lucid. The *MGFs* of the above distributions are also derived and parametric relations are obtained for certain basic properties of the distributions.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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