British Journal of Mathematics & Computer Science
12(5): 1-16, 2016, Article no.BJMCS.21867
ISSN: 2231-0851

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On Cyclic Associative Abel-Grassman Groupoids

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Article Information

DOI: 10.9734/BJMCS/2016/21867
Editor(s):
(1) Zuomao Yan, Department of Mathematics, Hexi University, China.
Reviewers:
(1) Francisco Bulnes, Tecnolgico de Estudios Superiores de Chalco, Mexico.
(2) Ali Mutlu, Celal Bayar University, Turkey.
(3) Ruben Dario Ortiz Ortiz, University of Cartagena, Colombia.
Complete Peer review History: http://sciencedomain.org/review-history/12138

Original Research Article

Received: 06 September 2015
Accepted: 05 October 2015
Published: 07 November 2015

Abstract

A new subclass of AG-groupoids, so called, cyclic associative Abel-Grassman groupoids or CA-AG-groupoid is studied. These have been enumerated up to order 6. A test for the verification of cyclic associativity for an arbitrary AG-groupoid has been introduced. Various properties of CA-AG-groupoids have been studied. Relationship among CA-AG-groupoids and other subclasses of AG-groupoids is investigated. It is shown that the subclass of CA-AG-groupoid is different from that of the AG*-groupoid as well as AG**-groupoids.

Keywords: AG-groupoid; cyclic associativity; CA-AG-groupoid; CA-test; Nuclear square; right alternative; bi-commutative; paramedial AG-groupoids.

2010 Mathematics Subject Classification: 20N05; 20N02; 20N99.

1 Introduction

An algebraic structure satisfying the left invertive law is called an Abel-Grassmann's groupoid (or simply AG-groupoid [1]). Other names for the same structure are left almost semigroup (LA-semigroup) [2], left invertive groupoid [3] and right modular groupoid [4]. Many authors have studied these structures and their properties. Many intersting properties of LA semigroups have been studied in [5] and the same authors have introduced the concept of locally associative LA semigroups

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in [6]. Paramedial groupoids have been studied in [4]. Many different aspect of AG-groupoids have been studied in [1, 7, 8, 9, 10, 11]. AG*-groupoids and AG**-groupoids are two different subclasses of AG-groupoid and their properties have been studied in the following [12, 13, 14].

This paper has been arranged as the following. In Section 3, notion of cyclic associative Abel-Grassman groupoid (CA-AG-groupoid) is introduced. These have been enumerated up to order 6. In section 4, CA-test for the verification of cyclic associativity of an AG-groupoid is given. Section 5, is devoted to the study of various properties of CA-AG-groupoids and their relations with different known subclasses of AG-groupoids. It is shown in section 6, the subclass of CA-AG-groupoids is distinct from both of the AG*-groupoids and AG**-groupoids.

2 Preliminaries

In this section some notions about Abel-Grassmann's groupoids are given. A groupoid $(S, \cdot)$ or simply S satisfying the left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in S$ is called an Abel-Grassmann's groupoid (or simply AG-groupoid [1, 2]). Through out this paper an AG-groupoid will be denoted by S, unless stated otherwise. S always satisfies the medial law: $ab \cdot cd = ac \cdot bd$ [15], while S with left identity e always satisfies the paramedial law: $ab \cdot cd = db \cdot ca$ [15]. A groupoid G is called right AG-groupoid or right almost semigroup (RA-semigroup) [2] if $a(bc) = c(ba)\, \forall a, b, c \in G$.

(a) S is called left nuclear square [16] if $\forall\, a, b, c \in S$, $a^2(bc) = (a^2b)c$, middle nuclear square if $a(b^2c) = (ab^2)c$, right nuclear square if $a(bc^2) = (ab)c^2$. S is called nuclear square [16] if it is left, middle and right nuclear square.

(b) S is called AG* [12], if $(ab)c = b(ac)$ for all $a, b, c \in S$.

(c) S is called AG** [13], if $a(bc) = b(ac)$ for all $a, b, c \in S$.

(d) S is called T^1-AG-groupoid [17] if for all $a, b, c, d \in S$, $ab = cd$ implies $ba = dc$.

(e) S is called left T^3-AG-groupoid (T^3_l-AG-groupoid) if for all $a, b, c \in S$, $ab = ac$ implies $ba = ca$ and is called right T^3-AG-groupoid (T^3_r-AG-groupoid) if $ba = ca$ implies $ab = ac$. S is called T^3-AG-groupoid [17], if it is both T^3_l and T^3_r.

(f) S is called Bol* [16] if it satisfies the identity $a(bc \cdot d) = (ab \cdot c)d$ for all $a, b, c, d \in S$.

(g) S is called left alternative if for all $a, b \in S$, $aa \cdot b = a \cdot ab$ and is called right alternative if $b \cdot aa = ba \cdot a$. S is called alternative [17], if it is both left alternative and right alternative.

(h) S is called right commutative if for all $a, b, c \in S$, $a(bc) = a(cb)$ and is called left commutative if $(ab)c = (ba)c$. S is called bi-commutative AG-groupoid [18], if it is right and left commutative.

An element $a \in S$ is called idempotent if $a^2 = a$ and an AG-groupoid having all elements as idempotent is called AG-2-band (simply AG-band) [8]. A commutative AG-band is called a semi-lattice. An AG-groupoid in which $(aa)a = a(aa) = a$ holds $\forall\, a \in S$ is called an AG-3-band [8]. An element $a \in S$ is left cancellative (respectively right cancellative) [16] if $\forall x, y \in S$, $ax = ay \Rightarrow x = y$ ($xa = ya \Rightarrow x = y$). An element is cancellative if it is both left and right cancellative. S is left cancellative (right cancellative, cancellative) if every element of S is left cancellative (right cancellative, cancellative). S may have all, some or none of its elements as cancellative [7]. S is called an AG-monoid, if it contains left identity e such that $ea = a\ \forall\, a \in S$. As AG-groupoid is a non-associative structure, thus left identity does not implies right identity and hence the identity.

3 CA-AG-groupoids

In this section a new subclass of AG-groupoids is being introduced. First existence of this class is shown. It is interesting to see that all AG-groupoids of order 2 are CA-AG-groupoids. There are

20 AG-groupoids of order 3, out of which only 12 are CA-AG-groupoids. Up to order 6, there are 9068 CA-AG-groupoids out of 40104513 AG-groupoids. A complete table up to order 6 is presented in this section.

Definition 1. *An AG-groupoid S satisfying the identity $a(bc) = c(ab)$ $\forall a, b, c \in S$ is called cyclic associative AG-groupoid (or shortly CA-AG-groupoid).*

The following example depicts the existence of CA-AG-groupoid.

Example 1. *CA-AG-groupoid of lowest order is given in Table* 1.

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

Table 1

3.1 Enumeration of CA-AG-groupoids.

Enumeration and classification of various mathematical entries is a well worked area of pure mathematics. In abstract algebra the classification of algebraic structure is an important pre-requisite for their construction. The classification of finite simple groups is considered as one of the major intellectual achievement of the twentieth century. Enumeration results can be obtained by a variety of means like; combinatorial or algebraic consideration. Non-associative structures, quasigroups and loops have been enumerated up to size 11 using combinatorial consideration and bespoke exhaustive generation software. FINDER (Finite domain enumeration) has been used for enumeration of IP loops up to size 13. Associative structures, semigroups and monoids have been enumerated up to size 9 and 10 respectively by constraint satisfaction techniques implemented in the Minion constraint solver with bespoke symmetry breaking provided by the computer algebra system GAP [19]. The third author of this article has implemented the same techniques in collaboration with Distler (the author of [20]) to deal the enumeration of AG-groupoids using the constraint solving techniques developed for semigroups and monoids. Further, they provided a simple enumeration of the structures by the constraint solver and obtained a further division of the domain into a subclass of AG-groupoids using the computer algebra system GAP and were able to enumerate all AG-groupoids up to isomorphism up to size 6.

It is worth mentioning that the data presented in [21] have been verified by one of the reviewers of the said article with the help of Mace-4 and Isofilter as has been acknowledged in the said article. All this, validate the enumeration and classification results of our CA-AG-groupoids, a subclass of AG-groupoids, as the same techniques and relevant data of [21] has been used for the purpose. Enumeration and classification of CA-AG-groupoids up to order 6 is given in Table 2.

Order	2	3	4	5	6
AG-groupoids	3	20	331	31913	40104513
CA-AG-groupoids	3	12	64	491	9068
Associative AG-groupoids	3	12	62	446	7510
Non-associative AG-groupoids	0	8	269	31467	40097003
CA, Non-associative	0	0	2	45	1565
Associative, Non-CA	0	0	0	0	07
CA and Associative	0	12	62	446	7503
Associative & non-commutative, CA	0	0	4	121	5360

Table 2. Enumeration of CA-AG-groupoids up to order 6

4 CA-Test for an AG-groupoid

Now a test for verification of cyclic associativity for an arbitrary AG-groupoid is suggested. For this define the following binary operations.

$$\begin{aligned} a \star b &= a(bx) \\ a \circ b &= x(ab), \text{ for some fixed } x \in S \end{aligned}$$

To test whether an arbitrary AG-groupoid $(S, \cdot)$ is cyclic associative, it is sufficient to check that $a \star b = a \circ b$ for all $x \in S$. To construct $\star$ table for any fixed $x \in S$, we rewrite x-column of the " $\cdot$ " table as index row of the $\star$ table and then multiply its elements from the left by the elements of the index column of the " $\cdot$ " table to obtain respective rows of the $\star$ table for x. The table of the operation $\circ$ for any fixed $x \in S$ is obtained by multiplying elements of the " $\cdot$ " table by x from the left. If the tables of operations $\star$ and $\circ$ coincide for all x in S, then $\star$ coincides with $\circ$, and thus $a(bc) = c(ab)$ and equivalently the AG-groupoid is cyclic associative. It is convenient to write $\circ$ tables under the $\star$ tables. We illustrate the procedure in the following example.

Example 2. *Consider the AG-groupoid in the Table* 3.

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	3

Table 3

To check cyclic associativity in the given Cayley's tables, we extend these tables in the way as described above. The upper tables to the right of the original " $\cdot$ " table are constructed for the operation $\star$, while the lower tables are for the operation $\circ$.

$\cdot$	1	2	3	4	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	3
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	1	2	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
					1				2				3				4			
					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2

Extended Table (3)

From the extended table (3), it is clear that the upper and lower tables on the right side of the original table coincide for all x in S, thus the AG-groupoid in Table 3 is a CA-AG-groupoid. The following example establishes this test fail in case of AG-groupoids which are not CA-AG-groupoids.

Example 3. *Consider the AG-groupoid as shown in the following Table* 4.

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Table 4

To check cyclic associativity in the given Cayley's tables, we extend these tables in the way as described above. The upper tables to the right of the original " · " table are constructed for the operation $\star$, while the lower tables are for the operation $\circ$.

$\cdot$	1	2	3	1	1	1	1	1	2	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	1	2	1	1	1	1	1	1	2	1	1	1
					1			2			3	
				1	1	1	1	1	1	1	1	1
				1	1	1	1	1	1	1	1	1
				1	1	1	1	1	1	1	2	1

Extended Table (4)

As in extended table (4), upper tables on the right hand side do not coincide with the respective lower tables, thus AG-groupoid given in Table 4 is not a CA-AG-groupoid.

5 Various Properties of CA-AG-groupoids

In section 3, existence of CA-AG-groupoid has been established. Various results are given to demonstrate that the subclass of CA-AG-groupoids is different from some very well known subclasses of groupoids. To begin with, following examples show that a CA-AG-groupoid may not be a semigroup.

Example 4. *Consider the algebraic structure given in Table* 5.

$\cdot$	1	2	3	4
1	2	3	3	3
2	4	3	3	3
3	3	3	3	3
4	3	3	3	3

Table 5

It is easy to verify that it is a CA-AG-groupoid, which is actually a non-associative CA-AG-groupoid of lowest order.

Although Table 2 shows that up to order 5 every associative AG-groupoid is CA, but the next example shows that a semigroup may not be a CA-AG-groupoid.

Example 5. *Let* $M_2 = \{(a_{ij}) \mid a_{ij} \in \mathbb{Z}\}$*, be the set of all* 2×2 *matrices with entries from* $\mathbb{Z}$*.* M_2 *is a semigroup under multiplication, but* M_2 *is not an AG-groupoid. Since* $(AB)C \neq (CB)A$ *for* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = B$ *and* $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *in* M_2*.*

However, the following is obvious.

Proposition 1. *Every commutative semigroup is a CA-AG-groupoid.*

Remark 1. *Every commutative AG-groupoid is associative.*

Corollary 1. *Every commutative AG-groupoid is a CA-AG-groupoid.*

In the following it is observed that subclass of Bol*-AG-groupoids is distinct from that of CA-AG-groupoids and we have the following. Recall that S is called a Bol*-AG-groupoid if it satisfies the identity $a(bc \cdot d) = (ab \cdot c)d \; \forall a, b, c, d \in S$.

Theorem 1. *Every CA-AG-groupoid is Bol*-AG-groupoid.*

Proof. Let S be a CA-AG-groupoid and $a, b, c, d \in S$, then by using cyclic associativity, left invertive law and medial law;

$$\begin{aligned} a(bc \cdot d) &= d(a \cdot bc) = d(c \cdot ab) = ab \cdot dc = c(ab \cdot d) = c(db \cdot a) \\ &= a(c \cdot db) = db \cdot ac = da \cdot bc = c(da \cdot b) = c(ba \cdot d) \\ &= d(c \cdot ba) = d(a \cdot cb) = d(b \cdot ac) = ac \cdot db = b(ac \cdot d) \\ &= b(dc \cdot a) = a(b \cdot dc) = dc \cdot ab = (ab \cdot c)d \\ \Rightarrow a(bc \cdot d) &= (ab \cdot c)d. \end{aligned}$$

Hence S is a Bol*-AG-groupoid. □

Following example shows that the converse of the Theorem 1, is not true.

Example 6. *Consider the Bol*-AG-groupoid of order* 3 *given in Table* 6.

·	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table 6

which is not a CA-AG-groupoid, because $1(2 \cdot 3) \neq 3(1 \cdot 2)$.

Further, we have the following.

Lemma 1. *Every Bol*-AG-band is commutative semigroup.*

Proof. Let S be a Bol*-AG-band and $a, b \in S$. Then

$$\begin{aligned} ab &= (aa)b = (ba)a = (bb \cdot a)a = (ab \cdot b)a = a(bb \cdot a) = a(ab \cdot b) \\ &= (aa \cdot b)b = bb \cdot aa = b^2a^2 = ba \Rightarrow ab = ba. \end{aligned}$$

Thus S is commutative and hence the result follows by Remark 1. □

From Lemma 1 and Proposition 1, it is obtained;

Corollary 2. *Every Bol*-AG-band is CA-AG-groupoid.*

Theorem 2. *Every CA-AG-band is commutative semigroup.*

Proof. Let S be a CA-AG-band and $a, b \in S$. Then

$$\begin{aligned} ab &= (aa)b = (ba)a = (bb \cdot a)a = (ab \cdot b)a \\ &= ab \cdot ab = aa \cdot bb = b(aa \cdot b) = b(b \cdot aa) \\ &= b(a \cdot ba) = ba \cdot ba = bb \cdot aa = ba. \quad \square \end{aligned}$$

As every CA-AG-groupoid is Bol*-AG-groupoid (by Theorem 1) and every Bol*-AG-groupoid is paramedial [16, Lemma 9], so the following corollary is obvious:

Corollary 3. *Every CA-AG-groupoid is paramedial.*

The following example shows that the converse of Corollary 3 is not valid.

Example 7. *Table* 7 *represents paramedial AG-groupoid of order* 3*, which is not a CA-AG-groupoid. as* $c(cb) \neq b(cc)$.

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	a

Table 7

It is proved in [16] that every paramedial AG-band is commutative semigroup. Also by Proposition 1, every commutative semigroup is CA-AG-groupoid. Hence we have the following.

Corollary 4. *Every paramedial AG-band is CA-AG-groupoid.*

Now relation of CA-AG-groupoid with left, middle and right nuclear square AG-groupoids is being studied in the following.

Theorem 3. *Let S be a CA-AG-groupoid. Then the following hold:*

(i) S is left nuclear square.

(ii) S is right nuclear square.

Proof. Let S be a CA-AG-groupoid and $a, b, c \in S$. Then

(i) By Corollary 3 and left invertive law, we have

$$a^2(bc) = (aa)(bc) = (ca)(ba) = (ba \cdot a)c = (aa \cdot b)c = (a^2b)c.$$

Hence S is left nuclear square.

(ii) By left invertive law, Part (i), cyclic associativity and Corollary 3.

$$\begin{aligned} (ab)c^2 &= (c^2b)a = c^2(ba) = a(c^2b) = a(cc \cdot b) = a(bc \cdot c) = c(a \cdot bc) \\ &= bc \cdot ca = ac \cdot cb = b(ac \cdot c) = b(cc \cdot a) = a(b \cdot cc) = a(bc^2). \end{aligned}$$

Hence S is right nuclear square. $\square$

Following example shows that neither a left nuclear square nor a right nuclear square AG-groupoid is always a CA-AG-groupoid.

Example 8. *Left nuclear square AG-groupoid of order* 3 *is given in Table* 8*, which is not a CA-AG-groupoid, because* $3(3 \cdot 2) \neq 2(3 \cdot 3)$.

·	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Table 8

In Table 9 a right nuclear square AG-groupoid of order 3 is given which is not a CA-AG-groupoid because $3(3 \cdot 2) \neq 2(3 \cdot 3)$.

·	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Table 9

Next attention is paid towards middle nuclear square AG-groupoid. Generally speaking no relation exist between middle nuclear square AG-groupoids and CA-AG-groupoid. The following example shows that neither every middle nuclear square AG-groupoid is CA-AG-groupoid, nor every CA-AG-groupoid is middle nuclear square AG-groupoid .

Example 9. *The following AG-groupoid of order* 3 *given in Table* 10*, is middle nuclear square AG-groupoid but not CA, because* $c(cb) \neq b(cc)$.

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	b

Table 10

CA-AG-groupoid of order 5, given in Table 11, is not a middle nuclear square AG-groupoid since $5(5^2 \cdot 5) \neq (5 \cdot 5^2)5$.

·	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	2
4	1	1	1	1	2
5	1	1	1	3	4

Table 11

However, a left alternative CA-AG-groupoid is middle nuclear square. As

Theorem 4. *Every left alternative CA-AG-groupoid is middle nuclear square.*

Proof. Let S be a left alternative CA-AG-groupoid and $a, b, c \in S$. Then by definition of left alternative, cyclic associativity, Corollary 3 and left invertive law;

$$\begin{aligned} a(b^2c) &= a(bb \cdot c) = a(b \cdot bc) = bc \cdot ab = (ab \cdot c)b = (cb \cdot a)b \\ &= ba \cdot cb = b(ba \cdot c) = b(ca \cdot b) = b(b \cdot ca) = ca \cdot bb \\ &= (bb \cdot a)c = (b \cdot ba)c = (a \cdot bb)c = (ab^2)c. \end{aligned}$$

Hence S is middle nuclear square. □

By coupling Theorem 3 and Theorem 4, the following corollary is obvious.

Corollary 5. *Every left alternative CA-AG-groupoid is nuclear square.*

As, for a right alternative AG-groupoid S the following conditions are equivalent [9, Theorem 3.2].

(i) S is right nuclear square.

(ii) S is middle nuclear square.

(iii) S is nuclear square .

Using these results and Theorem 3, we conclude that:

Corollary 6. *Every right alternative and hence alternative CA-AG-groupoid is nuclear square.*

6 CA-AG-Groupoids, AG* and AG**-groupoids

In this section it is seen that in general CA-AG-groupoid are neither AG*-groupoids nor AG**-groupoids. Thus the subclass, CA-AG-groupoid is different from both of AG*-groupoids and AG**-groupoids. To begin with, consider the following;

Example 10. *CA-AG-groupoid of order* 4*, given in Table* 12*, is not an AG*-groupoid because* $(aa)a \neq a(aa)$.

$\cdot$	a	b	c	d
a	b	c	c	c
b	d	c	c	c
c	c	c	c	c
d	c	c	c	c

Table 12

CA-AG-groupoid of order 8, presented in Table 13, is not an AG**-groupoid because $3(2\cdot 1) \neq 2(3\cdot 1)$

$\cdot$	1	2	3	4	5	6	7	8
1	4	4	6	4	4	4	8	4
2	5	4	4	4	4	8	4	4
3	4	7	4	4	8	4	4	4
4	4	4	4	4	4	4	4	4
5	4	4	4	4	4	4	4	4
6	4	4	4	4	4	4	4	4
7	4	4	4	4	4	4	4	4
8	4	4	4	4	4	4	4	4

Table 13

Further the following example establishes that there are AG**-groupoids which are not CA-AG-groupoid.

Example 11. *AG**-groupoid of order* 3*, given in Table* 14*, is not a CA-AG-groupoid because* $2(2\cdot 3) \neq 3(2\cdot 2)$.

·	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Table 14

However, we have the following;

Theorem 5. *Every right commutative AG**-groupoid is a CA-AG-groupoid.*

Proof. Let S be a right commutative AG**-groupoid and $a, b, c \in S$. Using definition of right commutative and AG**-groupoid, we have $a(bc) = a(cb) = c(ab)$. Hence S is a CA-AG-groupoid. □

The following example is to show that above Theorem is true only for AG**-groupoid.

Example 12. *Right commutative AG-groupoid of order* 3*, given in Table* 15*, is not a CA-AG-groupoid because* $3(3 \cdot 1) \neq 1(3 \cdot 3)$.

·	1	2	3
1	1	1	1
2	1	1	1
3	2	2	2

Table 15

The following example is to show that there are right commutative AG-groupoids which are not AG**-groupoids

Example 13. *Table* 16*, represents a right commutative AG-groupoid of order* 3*, which is not an AG**-groupoid because* $a(cb) \neq c(ab)$.

·	a	b	c
a	a	a	a
b	a	a	a
c	b	b	b

Table 16

The following example is to show that neither every CA-AG-groupoid is right commutative, nor every AG** is right commutative.

Example 14. *Table 13, given above represents a CA-AG-groupoid of order* 8*. As* $1(2 \cdot 3) \neq 1(3 \cdot 2)$*, so it is not a right commutative AG-groupoid. Table* 17 *represents an AG**-groupoid of order* 3*, which is not right commutative because* $3(2 \cdot 3) \neq 3(3 \cdot 2)$.

·	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Table 17

However, we have the following;

Theorem 6. *Let S be a CA-AG-groupoid then S is right commutative if and only if S is AG**.*

Proof. Let S be a CA-AG-groupoid and $a, b, c \in S$. Suppose S is right commutative, then, by using definition of right commutative and cyclic associativity, we have $a(bc) = a(cb) = b(ac)$. Hence S is AG**. Conversely, let S be an AG**-groupoid, then by cyclic associativity and definition of AG**, we have

$$a(bc) = c(ab) = a(cb) = b(ac) = c(ba) = a(cb) \Rightarrow a(bc) = a(cb).$$

Hence S is right commutative. □

As every AG** having a cancellative element is T^1-AG-groupoid [10] and hence is T^3-AG-groupoid [10], so we have the following corollary.

Corollary 7. *Every right commutative CA-AG-groupoid having a cancellative element is a T^1-AG-groupoid and hence a T^3-AG-groupoid.*

Since every T^1-AG-groupoid is AG** [9, Theorem 3.4], thus;

Corollary 8. *CA-T^1-AG-groupoid is right commutative.*

Theorem 7. *An AG*-band is CA.*

Proof. Let S be an AG*-band and $a, b, c \in S$. Then

$$\begin{aligned} a(bc) &= (ba)c = (ca)b = (ca)(bb) = (cb)(ab) = (a \cdot cb)b = (b \cdot cb)a \\ &= (cb \cdot b)a = (bb \cdot c)a = (bc)a = (ac)b = c(ab) \Rightarrow a(bc) = c(ab). \end{aligned}$$

Hence S is CA-AG-groupoid. □

Following example verifies that the converse of Theorem 7, is not valid.

Example 15. *Table 18, represents a CA-AG-groupoid of order 4, which is not AG*-band, because $(4 \cdot 4)4 \neq 4(4 \cdot 4)$ and $2 \cdot 2 \neq 2$.*

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	3

Table 18

Lemma 2. *An AG*-band is a semigroup.*

Proof. Let S be an AG*-band and $a, b, c \in S$. Then

$$\begin{aligned} a(bc) &= (ba)c = (ba)(cc) = (bc)(ac) = (a \cdot bc)c = (c \cdot bc)a \\ &= (bc \cdot c)a = (cc \cdot b)a = (cb)a = (ab)c \Rightarrow a(bc) = (ab)c. \end{aligned}$$

Hence S is a semigroup. □

Since every AG**-band is a commutative semigroup [16, Lemma 20] and every AG**-3-band is a commutative semigroup [13], also by Proposition 1, every commutative semigroup is CA-AG-groupoid. Hence we have the following corollary.

Corollary 9. (*i*) *Every AG**-band is a CA-AG-groupoid.* (*ii*) *Every AG**-3-band is a CA-AG-groupoid.*

As every AG-monoid is AG** [13], so from Corollary 9 the following is obvious;

Corollary 10. *Every AG-band having left identity is CA-AG-groupoid.*

Theorem 8. *Every CA-AG-3-band is a commutative semigroup.*

Proof. Let S be a CA-AG-3-band. Then $\forall\, a, b \in S$, by definition of AG-3-band, left invertive law and Corollary 3;

$$\begin{aligned} ab &= (aa \cdot a)b = (ba)(aa) = ((bb \cdot b)a)(aa) \\ &= (aa \cdot a)(bb \cdot b) = (aa \cdot bb)(a \cdot b) = (b \cdot bb)(a \cdot aa) \\ \Rightarrow ab &= ba. \end{aligned}$$

Thus S is commutative and hence is a commutative semigroup. □

Remark 2. *Let denote the set of all idempotents of an AG-groupoid S by $E(S)$.*

Lemma 3. *For a CA-AG-groupoid S, if $E(S) \neq \phi$ then $E(S)$ is a semi-lattice.*

Proof. Let S be a CA-AG-groupoid and a, $b \in E(S)$. Then

$$\begin{aligned} ab &= (aa)(bb) = (ab)(ab) = b(ab \cdot a) = a(b \cdot ab) \\ &= a(b \cdot ba) = a(a \cdot bb) = (bb)(aa) = ba. \end{aligned}$$

Thus $E(S)$ is commutative. Hence $E(S)$ is a commutative semigroup of idempotents and thus is a semi-lattice. □

Lemma 4. *Let S be a CA-AG-groupoid such that $E(S) \neq \phi$. Then e is the identity of eS and Se for every $e \in E(S)$.*

Proof. (i) Using cyclic associativity, left invertive law, and Corollary 3;

$$\begin{aligned} e(eS) &= \bigcup_{a\in S} e(ea) = \bigcup_{a\in S}(ee)(ea) = \bigcup_{a\in S}(ae)(ee) \\ &= \bigcup_{a\in S}(ae)e = \bigcup_{a\in S}(ee)a = \bigcup_{a\in S} ea = eS. \end{aligned}$$

Thus e is the left identity for eS. Also by left invertive law, medial law, cyclic associativity and Corollary 3;

$$\begin{aligned} (eS)e &= \bigcup_{a\in S}(ea)e = \bigcup_{a\in S}(ee \cdot a)e = \bigcup_{a\in S}(ea)(ee) = \bigcup_{a\in S}(ee)(ae) \\ &= \bigcup_{a\in S} e(ae) = \bigcup_{a\in S} e(ea) = \bigcup_{a\in S}(ee)(ea) = \bigcup_{a\in S}(ae)(ee) \\ &= \bigcup_{a\in S}(ae)e = \bigcup_{a\in S}(ee)a = \bigcup_{a\in S} ea = eS. \end{aligned}$$

Thus e is also right identity of eS and hence the identity of eS.

(ii) By using cyclic associativity;

$$e(Se) = \bigcup_{a\in S} e(ae) = \bigcup_{a\in S} e(ea) = \bigcup_{a\in S} a(ee) = \bigcup_{a\in S} ae = Se.$$

Thus e is the left identity for Se. Also, by Corollary 3 and cyclic associativity;

$$\begin{aligned}(Se)e &= \bigcup_{a\in S}(ae)e = \bigcup_{a\in S}(ae)(ee) = \bigcup_{a\in S}(ee)(ea) \\ &= \bigcup_{a\in S} e(ea) = \bigcup_{a\in S} a(ee) = \bigcup_{a\in S} ae = Se.\end{aligned}$$

Thus e is also the right identity for Se. Hence e is the identity for Se.

□

The following example shows that there exist left commutative AG-groupoids which are not CA-AG-groupoids.

Example 16. *Left commutative AG-groupoid of order* 3 *is presented in Table 19, which is not a CA-AG-groupoid because* $3(2\cdot 1) \neq 1(3\cdot 2)$.

·	1	2	3
1	1	1	1
2	1	1	1
3	2	2	1

Table 19

Next, we have the following;

Theorem 9. *Every left commutative AG*-groupoid is a CA-AG-groupoid.*

Proof. Let S be a left commutative AG*-groupoid and $a, b, c \in S$, then by definition of AG*, left commutative and left invertive law;

$$a(bc) = (ba)c = (ca)b = (ac)b = c(ab) \Rightarrow a(bc) = c(ab).$$

Hence S is CA-AG-groupoid. □

Following example shows that a left commutative CA-AG-groupoid may not be an AG*-groupoid.

Example 17. *Left commutative CA-AG-groupoid of order* 4*, is given in Table* 20*, which is not AG*-groupoid because* $(d\cdot d)d \neq d(d\cdot d)$.

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	a	c

Table 20

Theorem 10. *Let S be a CA-AG*-groupoid. Then the following holds.*

(i) S is bi-commutative.
(ii) S is semigroup.

Proof. (i) Let S be a CA-AG*-groupoid and $a, b, c \in S$, then

$$\begin{aligned}(ab)c &= (cb)a = b(ca) = a(bc) = c(ab) = (ac)b = (bc)a \\ &= c(ba) = a(cb) = (ca)b = (ba)c \Rightarrow (ab)c = (ba)c.\end{aligned}$$

Hence S is left commutative. Again

$$\begin{aligned}a(bc) &= c(ab) = b(ca) = (cb)a = (ab)c = b(ac) \\ &= c(ba) = a(cb) \Rightarrow a(bc) = a(cb).\end{aligned}$$

Thus S is also right commutative and hence S is bi-commutative.

(ii) Let S be a CA-AG*-groupoid and $a, b, c \in S$. Then

$$a(bc) = c(ab) = b(ca) = (cb)a = (ab)c \Rightarrow a(bc) = (ab)c.$$

Hence S is a semigroup. □

The following example is to show that there exist bi-commutative AG-groupoids which are neither CA-AG-groupoids nor AG*-groupoids.

Example 18. *Table 21 represents a bi-commutative AG-groupoid of order 4, which is neither CA-AG-groupoid nor AG*-groupoid.*

·	1	2	3	4
1	1	1	3	3
2	1	1	4	4
3	3	3	1	1
4	3	3	1	1

Table 21

Next example shows that there exist CA-AG*-groupoids which are not bi-commutative AG-groupoid.

Example 19. *Table 22 shows a CA-AG*-groupoid of order 4, which is not commutative.*

·	1	2	3	4
1	3	3	3	3
2	4	3	3	3
3	3	3	3	3
4	3	3	3	3

Table 22

From the above discussion we conclude that:

Corollary 11. *CA-AG*-groupoid is a non-commutative semigroup.*

Since every T^1-AG-3-band is AG*-groupoid [14], so from Theorem 10, we have the following corollary.

Corollary 12. (*i*) *Every CA-T^1-AG-3-band is a bi-commutative AG-groupoid.*
(*ii*) *Every CA-T^1-AG-3-band is a semigroup.*

The relationship among different subclasses of AG-groupoids is given in the following Venn Diagram.

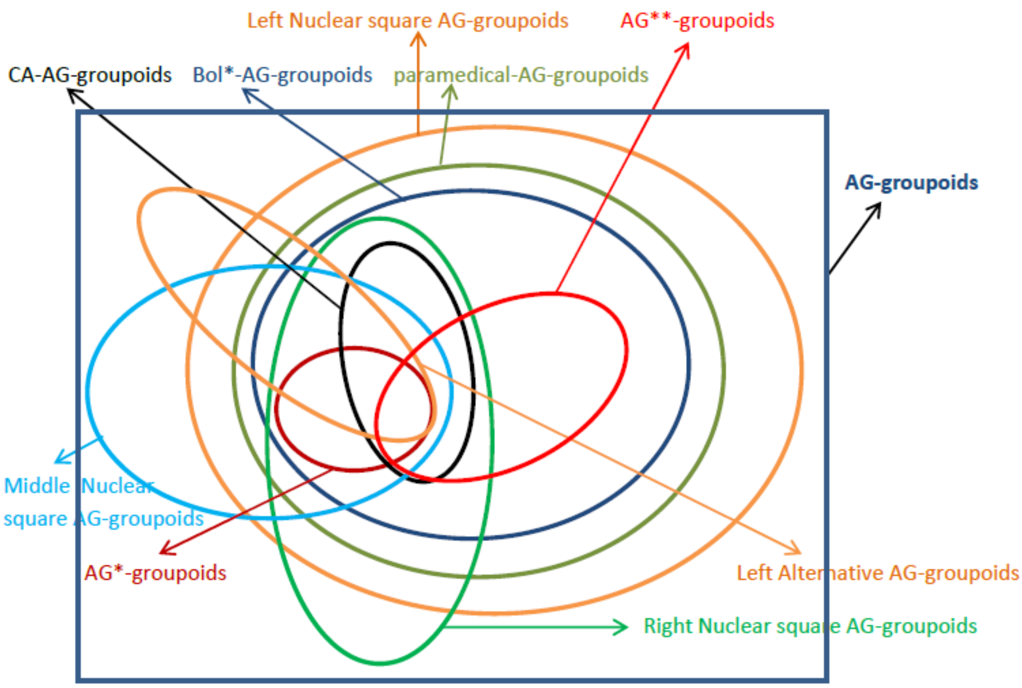

Fig. 1. Relations among different subclasses of AG-groupoids

7 Conclusion

We introduced CA-AG-groupoid as a new subclass of AG-groupoids. We used the modern computational techniques of GAP, Prover-9 and Mace-4 to enumerate it up to order 6. We produced sufficient counterexamples and provided several other examples to improve the standard of this article. We presented CA-test for verification of cyclic associativity in arbitrary AG-groupoid and precisely discussed some fundamental properties of CA-AG-groupoids and established its relations with other subclasses of AG-groupoids and with semigroup.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Protic PV, Stevanovic N. On Abel-grassmann's groupoids (review). Proceeding of Mathematics Conference in Pristina. 1994;31-38.

[2] Kazim MA, Naseeruddin M. On almost semigroups. Portugaliae Mathematica. 1972;2:1-7.

[3] Holgate P. Groupoids satisfying a simple invertive law.Mathematics Student. 1992;61:101-106.

[4] Cho JR, Pusan, Jezek J, Kepka T. Paramedial groupoids. Czechoslovak Mathematical Journal. 1996;49(124):277-290.

[5] Mushtaq Q, Yusuf SM. On LA-Semigroups. The Aligarh Bulitun Mathematics. 1978;8:65-70.

[6] Mushtaq Q, Yusuf SM. On locally associative LA-Semigroups. J. Nat. Sci. Maths. 1979;19(1):57-62.

[7] Shah M, Shah T, Ali A. On the cancellativity of ag-groupoids. International Mathematical Forum. 2011;6(44);2187-2194.

[8] Stevanovic N, Protic PV. Some decomposition on Abel-Grassmann's groupoids. PU. M. A. 1997;8:355-366.

[9] Rashad M, Ahmad I, Amanullah, Shah M. On relation between right alternative and nuclear square AG-groupoid. International Mathematical Forum. 2013; 8(5):237-243.

[10] Shah M, Ali A, Ahmad I. On introduction of new classes of AG-groupoids. Research Journal of Recent Sciences. 2013;2(1):67-70.

[11] Mushtaq Q, Khan M. Direct product of abel grassmann's groupoids. Journal of Interdisciplinary Mathematics. 2008;11(4):461-467.

[12] Mushtaq Q, Kamran MS. On LA-semigroups with weak associative law. Scientific Khyber.1989;1(11):69-71.

[13] Protic PV, Bozinovic M. Some congruences on an AG**-groupoid. Algebra Logic and Discrete Math. 1995;879-886.

[14] Ahmad I, Rashad M, Shah M. Some properties of AG*. Research Journal of Recent Science 2013;2(4):91-93.

[15] Jezek J, Kepka T. Medial Groupoids. Academia Nakladatelstvi Ceskoslovenske Akademie Ved; 1983.

[16] Shah M. A theoretical and computational investigation of AG-groups. PhD thesis. Quaid-i-Azam University Islamabad, Pakistan; 2012.

[17] Shah M, Ahmad I, Ali A. Discovery of new classes of AG-groupoids. Res. J. Recent Sci. 2012;1(11):47-49.

[18] Rashad M. Investigation and classification of new subclasses of AG-groupoids. PhD thesis, Malakand University, Pakistan; 2015.

[19] GAP: Groups Algorithm and Programming. 2013; Version 4.6.3.

[20] Distler A, Kelsey T. The monoids of order eight and nine and ten. Ann. Math. Artif. Intell. 2009;56(1):3-21.

[21] A. Distler, Shah M, Sorge V. Enumeration of AG-groupoids. Joint proceedings of the 18th Calculemus 2011 and MKM 2011, Vol. 6824 of Lecture Notes in Artificial Intellegence, Springer Verlag. 2014; $1-14$.

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