British Journal of Mathematics & Computer Science
12(5): 1-7, 2016, Article no.BJMCS.20641
ISSN: 2231-0851

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Improvement Results for Oscillatory Behavior of Second Order Neutral Differential Equations with Nonpositive Neutral Term

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Article Information

DOI: 10.9734/BJMCS/2016/20641
Editor(s):
(1) Dragos-Patru Covei, Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana, Romania.
Reviewers:
(1) Vasil G. Angelov, University of Mining and Geology St.I.Rilski, Bulgaria.
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Complete Peer review History: http://sciencedomain.org/review-history/12136

Original Research Article

Received: 01 August 2015
Accepted: 12 September 2015
Published: 07 November 2015

Abstract

In this paper we obtain new criteria for the oscillation of all solutions of second order neutral differential equations with nonpositive neutral term, which improve some of the results in [1]. Examples are provided to illustrate the main results.

Keywords: Oscillation; neutral differential equation; nonpositive neutral term.

2010 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

In this paper, we are concerned with a nonlinear neutral differential equation of the form

$$(r(t)(z'(t))^{\alpha})' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0 \geq 0 \tag{1.1}$$

where $z(t) = x(t) - a(t)x(\tau(t))$, and $\alpha > 0$ is a ratio of odd positive integers. Throughout, we assume that the following conditions are satisfied without further mention:

(C_1) $r, a, q \in C([t_0, \infty), \mathbb{R})$, $r(t) > 0$, $\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty$, $0 \leq a(t) < p < 1$, p is a constant, and $q(t) > 0$ for all $t \geq t_0$;

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(C_2) $\tau \in C([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$;

(C_3) $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma'(t) > 0$, $\sigma(t) \le t$, and $\lim_{t\to\infty} \sigma(t) = \infty$;

(C_4) $f \in C(\mathbb{R},\mathbb{R})$, $uf(u) > 0$ for all $u \neq 0$, and there exists a positive constant k such that $\frac{f(u)}{u^\alpha} \ge k$ for all $u \neq 0$.

By a solution of equation (1.1), we mean a continuous function $x \in ([T_x,\infty),\mathbb{R})$, $T_x \ge t_0$ which has the property $r(t)(z'(t))^\alpha \in C'([T_x,\infty),\mathbb{R})$ and satisfies equation (1.1) on the interval $[T_x,\infty)$. We consider only those solutions of equation (1.1) which satisfy condition $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$, and assume that equation (1.1) possess such solutions. As usual, a solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[T_x,\infty)$; otherwise it is said to be nonosicllatory.

In recent years, there has been an increasing interest in studying oscillatory and nonoscillatory behavior of solutions of different classes of differential equations due to the fact that they have many applications in science and engineering, see for example, [2] and [3]. In particular, lot of papers deal with oscillatory behavior of second order delay and neutral type differential equations, see, for instance, [4], [5], [6], [7], [1], [8], [9], [10], [11] and the references cited therein.

In [8], [12], the authors obtained several oscillation theorems for equation (1.1) under the assumptions that

$$0 \le a(t) \le a < 1 \tag{1.2}$$

and

$$\tau(t) = t - \tau_0 \le t \quad \text{and} \quad \sigma(t) = t - \sigma_0 \le t. \tag{1.3}$$

Recently in [1], the authors considered the equation (1.1) under the conditions (1.2) and $\int_{t_0}^\infty r^{-1/\alpha}(t)dt = \infty$, and established that all solutions of equation (1.1) are either oscillatory or tend to zero monotonically. Also, the same authors raised the question when all solutions are just oscillatory for the equation (1.1) when $\int_{t_0}^\infty r^{-1/\alpha}(t)dt = \infty$.

Motivated by the above observation, in this paper we obtain conditions for the oscillation of all solutions of equation (1.1). In Section 2, we present oscillation theorems for equation (1.1) and in Section 3, we provide some examples to illustrate the main results. Thus the results obtained in this paper improve that of in [1].

2 Oscillation Results

In this section, we present some new oscillation results for the equation (1.1). In the sequel, all functional inequalities are assumed to hold for all t large enough. Without loss of generality, we can deal only with positive solutions of equation (1.1).

Lemma 2.1. Assume that x is a positive solution of equation (1.1). Then z satisfies the following two possible cases:

(I) $z(t) > 0$, $z'(t) > 0$, $(r(t)(z'(t))^\alpha)' \le 0$;

(II) $z(t) < 0$, $z'(t) > 0$, $(r(t)(z'(t))^\alpha)' \le 0$.

Proof. The proof can be found in [1]. □

Lemma 2.2. If x is a positive solution of equation (1.1) such that Case(I) of Lemma 2.1 holds, then

$$x(t) \ge z(t) \ge R(t) r^{\frac{1}{\alpha}}(t) z'(t), \; t \ge T \ge t_0, \tag{2.1}$$

and $\frac{z(t)}{R(t)}$ is strictly decreasing, where $R(t) = \int_{t_0}^t r^{-\frac{1}{\alpha}}(s)ds$.

Proof. From the definition of z, we have $x(t) = z(t) + a(t)x(\tau(t))$ and therefore $x(t) \geq z(t)$ for all $t \geq T \geq t_0$. Since $r(t)(z'(t))^\alpha$ is nonincreasing, we have for $t \geq T \geq t_0$

$$z(t) = z(T) + \int_T^t \frac{(r(t)(z'(t))^\alpha)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)} ds \geq R(t) r^{\frac{1}{\alpha}}(t) z'(t).$$

Now

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{r^{\frac{1}{\alpha}}(t)R(t)z'(t) - z(t)}{r^{\frac{1}{\alpha}}(t)R^2(t)} \leq 0, \ t \geq T \geq t_0$$

by (2.1). Hence $\frac{z(t)}{R(t)}$ is nonincreasing for all $t \geq T \geq t_0$. This completes the proof. □

The following Theorems 2.3 and 2.5 are improving of Theorems 3.1 and 3.2 of [1] respectively.

Theorem 2.3. *Assume that $\sigma(t) < \tau(t)$ for all $t \geq t_0$. If there exists a positive nondecreasing function $\rho \in C'([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $T \geq t_0$*

$$\int_T^\infty \left[k\rho(t)q(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha - \frac{\rho'(t)(\sigma'(t))^\alpha}{R^\alpha(\sigma(t))}\right] dt = \infty, \tag{2.2}$$

and

$$\lim_{t\to\infty} sup \int_{\tau^{-1}(\sigma(t))}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(\int_s^t q(u)du\right)^{\frac{1}{\alpha}} ds > \frac{p}{k^{\frac{1}{\alpha}}}, \tag{2.3}$$

then every solution of equation (1.1) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of equation (1.1), since the proof for the negative case is similar. Then there exists a $T \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq T$. Then by Lemma 2.1 $z(t)$ satisfies one of the Cases (I) and (II) for all $t \geq T$.
Case (I). From the definition of z and (C_2), we have

$$x(t) \geq z(t) + a(t)z(\tau(t)) \geq \left(1 + a(t)\frac{R(\tau(t))}{R(t)}\right) z(t), \quad t \geq T, \tag{2.4}$$

where we have used $\frac{z(t)}{R(t)}$ is decreasing. Using (2.4) and (C_4) in equation (1.1), we have

$$\left(r(t)(z'(t))^\alpha\right)' + kq(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha z^\alpha(\sigma(t)) \leq 0, \ t \geq T.$$

Define

$$w(t) = \rho(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))}, \ t \geq T.$$

Then $w(t) > 0$ for $t \geq T$, and

$$w'(t) = \rho'(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\sigma(t))} + \rho(t)\frac{(r(t)(z'(t))^\alpha)'}{z^\alpha(\sigma(t))} - \rho(t)\alpha\frac{r(t)(z'(t))^\alpha}{z^{\alpha+1}(\sigma(t))}z'(\sigma(t))\sigma'(t).$$

Using $r(t)(z'(t))^\alpha \leq r(\sigma(t))(z'(\sigma(t)))^\alpha$ and (2.1) in the last inequality, we have

$$w'(t) \leq -k\rho(t)q(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha + \frac{\rho'(t)(\sigma'(t))^\alpha}{R^\alpha(\sigma(t))}, \ t \geq T.$$

Integrating the last inequality from T to t, we obtain

$$\int_T^t \left[k\rho(s)q(s)\left(1 + a(\sigma(s))\frac{R(\tau(\sigma(s)))}{R(\sigma(s))}\right)^\alpha - \frac{\rho'(s)(\sigma'(s))^\alpha}{R^\alpha(\sigma(s))}\right] ds \leq w(T)$$

which contradicts (2.2).

Case(II). From the definition of z and (C_1), we have

$$x(\tau(t)) > -\frac{z(t)}{p}, \quad t \geq T \geq t_0. \tag{2.5}$$

Using (2.5) and (C_4) in equation (1.1), we obtain

$$\big(r(t)(z'(t))^{\alpha}\big)' - \frac{k}{p^{\alpha}}q(t)z^{\alpha}(\tau^{-1}(\sigma(t))) \leq 0, \; t \geq T. \tag{2.6}$$

Integrating (2.6) from s to t for $t > s$, we have

$$r(t)(z'(t))^{\alpha} - r(s)(z'(s))^{\alpha} - \frac{k}{p^{\alpha}}\int_s^t q(u)z^{\alpha}(\tau^{-1}(\sigma(u)))du \leq 0.$$

Again integrating the last inequality from $\tau^{-1}(\sigma(t))$ to t for s, and using the fact that z is negative and increasing, we have

$$z(\tau^{-1}(\sigma(t))) - z(t) \leq \frac{k^{\frac{1}{\alpha}}}{p}z(\tau^{-1}(\sigma(t)))\int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)}\left(\int_s^t q(u)du\right)^{\frac{1}{\alpha}} ds$$

or

$$\frac{p}{k^{\frac{1}{\alpha}}} \geq \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)}\left(\int_s^t q(u)du\right)^{\frac{1}{\alpha}} ds$$

which contradicts (2.3). The proof is now completed. □

Let $\rho(t) = 1$. Then from Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let $\tau(t) < \sigma(t)$ for $t \geq t_0$. If condition* (2.3) *and*

$$\int_{t_0}^{\infty} q(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)dt = \infty, \tag{2.7}$$

are satisfied then every solution of equation (1.1) *is oscillatory.*

For $\alpha > 1$, we derive the following result different from Theorem 2.3.

Theorem 2.5. Let $\alpha > 1$ hold, and $\sigma(t) < \tau(t)$ for $t \geq t_0$. Assume that there exists a positive nondecreasing function $\rho \in C'([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large $T \geq t_0$,

$$\int_T^{\infty}\left[k\rho(t)q(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha} - \frac{(\rho'(t))^2 r^{\frac{1}{\alpha}}(\sigma(t))}{4\alpha\rho(t)\sigma'(t)R^{\alpha-1}(\sigma(t))}\right]dt = \infty. \tag{2.8}$$

If condition (2.3) holds, then every solution of equation (1.1) is oscillatory.

Proof. As above, we assume that x is a positive solution of equation (1.1). Then by Lemma 2.1, z satisfies one of (I) and (II). Assume first that z satisfies Case (I) of Lemma 2.1. Then define w as in the proof of Theorem 2.3. Then $w > 0$ and

$$w'(t) = -k\rho(t)q(t)\left(1 + a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha} + \frac{\rho'(t)}{\rho(t)}w(t) - \alpha\sigma'(t)w(t)\frac{z'(\sigma(t))}{z(\sigma(t))}. \tag{2.9}$$

Now by (2.1) and $r(t)(z'(t))^{\alpha} \leq r(\sigma(t))(z'(\sigma(t)))^{\alpha}$, we have

$$\frac{z'(\sigma(t))}{z(\sigma(t))} \geq \frac{R^{\alpha-1}(\sigma(t))(\sigma'(t))^{\alpha-1}}{r^{\frac{1}{\alpha}}(\sigma(t))\rho(t)}w(t), \; t \geq T. \tag{2.10}$$

Using (2.10) in (2.9), we obtain

$$\begin{aligned} w'(t) &\leq -k\rho(t)q(t)\left(1+a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha}+\frac{\rho'(t)}{\rho(t)}w(t)-\alpha\frac{R^{\alpha-1}(\sigma(t))(\sigma'(t))^{\alpha}}{r^{\frac{1}{\alpha}}(\sigma(t))\rho(t)}w^2(t) \\ &\leq -k\rho(t)q(t)\left(1+a(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha}+\frac{1}{4\alpha}\frac{(\rho'(t))^2 r^{\frac{1}{\alpha}}(\sigma(t))}{\rho(t)R^{\alpha-1}(\sigma(t))(\sigma'(t))^{\alpha}}. \end{aligned}$$

Integrating the last inequality from T to t, we obtain

$$\int_T^t\left[k\rho(s)q(s)\left(1+a(\sigma(s))\frac{R(\tau(\sigma(s)))}{R(\sigma(s))}\right)^{\alpha}-\frac{(\rho'(s))^2 r^{\frac{1}{\alpha}}(\sigma(s))}{4\alpha\rho(s)\sigma'(s)R^{\alpha-1}(\sigma(s))}\right]ds \leq w(T),$$

which contradicts (2.8).

If z satisfies Case (II) of Lemma 2.1, then proceeding as in the proof of Theorem 2.3 (Case(II)), we obtain a contradiction with (2.3). The proof is now completed. □

Next we consider the case $\alpha = 1,\ \tau(t) = t-k$, and $\sigma(t) = t-\ell$ where k and ℓ are positive constants with $\ell > k$.

Theorem 2.6. *Assume conditions $(C_1)-(C_4)$ hold with $\alpha = 1,\ \tau(t) = t-k$, and $\sigma(t) = t-\ell$ where k and ℓ are positive constants with $\ell > k$. If*

$$\lim_{t\to\infty}\inf\int_{t-\ell}^t q(s)(R(s-\ell)+a(s-\ell)R(s-\ell-k))ds > \frac{1}{ke}, \tag{2.11}$$

and

$$\lim_{t\to\infty}\sup\int_{t-\ell+k}^t \frac{1}{r(s)}\left(\int_s^t q(u)du\right)ds > \frac{p}{k}, \tag{2.12}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. As above, we assume that x is a positive solution of equation (1.1). Then by Lemma 2.1, z satisfies one of (I) and (II).

Case (I). Using (2.4) and (C_4) in equation (1.1), we have

$$(r(t)z'(t))' + kq(t)\left(1+a(t-\ell)\frac{R(t-\ell-k)}{R(t-\ell)}\right)z(t-\ell) \leq 0,\ t \geq T. \tag{2.13}$$

From Lemma 2.1, we have

$$z(t-\ell) \geq R(t-\ell)r(t-\ell)z(t-\ell), \quad t \geq T. \tag{2.14}$$

Using (2.14) in (2.13) we obtain

$$(r(t)z'(t))' + kq(t)(R(t-\ell)+a(t-\ell)R(t-\ell-k))r(t-\ell)z(t-\ell) \leq 0.$$

Let $w(t) = r(t)z'(t)$. Then $w(t) > 0$ and

$$w'(t) + kq(t)(R(t-\ell)+a(t-\ell)R(t-\ell-k))w(t-\ell) \leq 0. \tag{2.15}$$

In view of Theorem 6.4.2 [2], the condition (2.11) implies that the inequality (2.15) has no positive solution, which is a contradiction.

Case (II). The proof is similar to that of Theorem 2.3 and hence the details are omitted. This completes the proof. □

3 Examples

In this section, we present some examples to illustrate the main results obtained in the previous section.

Example 3.1. Consider a second order neutral differential equation

$$\left((z'(t))^{\frac{1}{3}}\right)' + \frac{4}{t}x^{\frac{1}{3}}\left(\frac{t}{3}\right) = 0, \quad t \geq 1, \tag{3.1}$$

where $z(t) = x(t) - \frac{1}{2}x(t/2)$. Here $\alpha = \frac{1}{3},\ r(t) = 1,\ q(t) = \frac{4}{t},\ \tau(t) = \frac{t}{2},\ \sigma(t) = \frac{t}{3}$, and $k = 1$. By taking $\rho(t) = 1$, we see that all conditions of Corollary 2.4 are satisfied and hence every solution of equation (3.1) is oscillatory.

Example 3.2. Consider a second order neutral differential equation

$$\left(t^2(z'(t))^3\right)' + \frac{10}{t^2}x^3\left(\frac{t}{3}\right) = 0, \quad t \geq 1, \tag{3.2}$$

where $z(t) = x(t) - \frac{1}{2}x(t/2)$. Here $\alpha = 3,\ r(t) = t^2,\ q(t) = \frac{10}{t^2},\ \tau(t) = \frac{t}{2},\ \sigma(t) = \frac{t}{3}$, and $k = 1$. By taking $\rho(t) = t$, we see that all conditions of Theorem 2.5 are satisfied and hence every solution of equation (3.2) is oscillatory.

Example 3.3. Consider a second order neutral differential equation

$$\left(x(t) - \frac{1}{2}x(t - \frac{\pi}{2})\right)'' + 8x(t - \pi) = 0, \quad t \geq 1. \tag{3.3}$$

It is easy to see that all conditions of Theorem 2.6 are satisfied and hence every solution of equation (3.3) is oscillatory. In fact $x(t) = sin4t$ is one such oscillatory solution of this equation.

4 Conclusions

This paper presents new criteria for the oscillation of all solutions of equation (1.1) under the condition $\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty$. The obtained results improve Theorems 3.1 and 3.2 of [1].

Acknowledgment

The authors grateful to all the referees for their helpful suggestions and comments.

Competing Interests

The authors declare that no competing interests exist.

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