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Full Length Research Paper

The half way series expansion

Alpha Ibrahim Turay

Department of Mathematics and Statistics, Fourah Bay College, University of Sierra Leone, Sierra Leone.

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A new formula for mathematics is derived which gives the upper half of the series expansion of the expression $(ay+b)^{2n}$, where n is a natural number. A proof of the new formula is given followed by a simple example to test its accuracy. This formula is helpful whenever n is large.

Key words: Series Expansion, fluid models, Intermolecular Potential model

INTRODUCTION

The usefulness of this formula is not yet known since it is new. Attempt was made to prove beyond reasonable doubt that it is possible to integrate $\int_\sigma^y \cdots \int_\sigma^y y^n dy_1 \cdots dy_n$ (this is, the function y^n is integrated n times, wherein the region of integration is $[\sigma \quad y]$, n can be any natural number, even if it turns out to be Avogadro's number, as long as the limits of integration is the same throughout. This problem surfaced while attempting to find a mathematically accurate solution to the standard thermodynamic equation for some fluid models. One of the intermolecular potential models attempted was that of the hard sphere which led to Equation 1. Earlier, it was perceived that it is not possible to perform this integration conveniently if n turns out to be a large number. Due to dedication and determination, a general solution was obtained which is true in general (Turay, 2018). That is,

$$\int_a^y \int_a^{y_1} \cdots \int_a^{y_2} y_1^n dy_1 dy_2 \ldots dy_n = \frac{n!}{(2n)!} a^n (y-a)^n \sum_{k=0}^{n} \binom{2n}{n-k} \left(\frac{y-a}{a}\right)^k \quad (1)$$

While finding a way of simplifying the aforementioned series, what can best be described as the halfway series expansion was obtained. The result of this series summation is pivotal to help simplify the result. But its other uses cannot be ascertained presently.

SERIES REPRESENTATION OF THE HALFWAY EXPANSION

Consider the function:

$$f(y) = (ay+b)^{2n} \quad (2)$$

The right hand side can be expanded and obtain (Burton, 2007; Guichar, 2017; Kalman, 1993):

$$(ay+b)^{2n} = (ay)^{2n}\binom{2n}{0} + (ay)^{2n-1}b\binom{2n}{1} + \cdots + (ay)^n b^n \binom{2n}{n} + \cdots + (ay)b^{2n-1}\binom{2n}{2n-1} + b^{2n}\binom{2n}{2n}$$

In series notation $(ay+b)^{2n} = \sum_{r=0}^{2n} \binom{2n}{r} (ay)^{2n-r} b^r$ (Burton,

E-mail: iituray@yahoo.com.

2007; Guichar, 2017), the upper halfway in the series expansion of the function $f(y)$ is given by

$$hf(y) = (ay)^n b^n \binom{2n}{n} + \cdots + (ay) b^{2n-1} \binom{2n}{2n-1} + b^{2n} \binom{2n}{2n} \qquad (3)$$

$$hf(y) = (ay)^n b^n \left[\binom{2n}{n} + \cdots + \left(\frac{b}{ay}\right)^{n-1} \binom{2n}{2n-1} + \left(\frac{b}{ay}\right)^n \binom{2n}{2n} \right] \qquad (4)$$

In series notation (Burton, 2007):

$$hf(y) = (ay)^n b^n \left[\binom{2n}{n} + \sum_{n-r=1}^{n} \left(\frac{b}{ay}\right)^{n-r} \binom{2n}{2n-r} \right]$$

This can be rewritten as

$$hf(y) = (ay)^n b^n \left[\sum_{n-r=0}^{n} \left(\frac{b}{ay}\right)^{n-r} \binom{2n}{2n-r} \right]$$

Now let $n - r = k$

Substituting this into the aforementioned equation gives (Guichar, 2017)

$$hf(y) = (ay)^n b^n \left[\sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k} \binom{2n}{n+k} \right] \qquad (5)$$

Thus, in series notation, the upper halfway expansion of $f(y) = (ay+b)^{2n}$ is given by (Burton, 2007; Guichar, 2017):

$$hf(y) = (ay)^n b^n \left[\sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k} \binom{2n}{n-k} \right] \qquad (6)$$

This holds since $\binom{2n}{n+k} = \binom{2n}{n-k}$

RIGOROUS PROOF OF THE SUMMATION OF SERIES

Furthermore, let us introduce the function:

$$h_1 f(y) = \sum_{k=0}^{n} \left(\frac{b}{ay}\right)^{k} \binom{2n}{n-k} \qquad (7)$$

Let us further make the substitution:

$$e^{2x} = \frac{b}{ay} \Rightarrow 2x = \ln\left(\frac{b}{ay}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{b}{ay}\right)$$

This reduces the function $h_1 f(y)$ to a new function $h_1 f(y) = H_1(x)$ (Wrede and Speigel, 2010):

where $H_1(x) = \sum_{k=0}^{n} \binom{2n}{n-k} e^{2kx} = \sum_{k=0}^{n} [\cosh 2kx + \sinh 2kx] \binom{2n}{n-k}$ (8)

$$= \sum_{k=0}^{n} \cosh 2kx \binom{2n}{n-k} + \sum_{k=0}^{n} \sinh 2kx \binom{2n}{n-k}$$

Introducing other new functions called:

$J(x) = \sum_{k=0}^{n} \binom{2n}{n-k} \cosh 2kx$ and $\phi(x) = \sum_{k=0}^{n} \binom{2n}{n-k} \sinh 2kx$

This gives

$$H_1(x) = J(x) + \phi(x) \qquad (9)$$

First attempt was made to prove that

$$J(x) = \sum_{k=0}^{n} \binom{2n}{n-k} \cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} + \frac{1}{2}\binom{2n}{n} \qquad (10)$$

This was started by finding an expansion for the function

$$= \binom{2n}{0} e^{2nx} + \binom{2n}{2n} e^{-2nx} + \binom{2n}{1} e^{2(n-1)x} + \binom{2n}{2n-1} e^{-2(n-1)x} + \cdots + \binom{2n}{p} e^{2(n-p)x} + \binom{2n}{2n-p} e^{-2(n-p)x} + \cdots + \binom{2n}{n}$$

$$\left(e^x + e^{-x}\right)^{2n} = \binom{2n}{0} e^{2nx} + \binom{2n}{1} e^{2(n-1)x} + \cdots + \binom{2n}{p} e^{2(n-p)x} + \cdots + \binom{2n}{2n-p} e^{-2(n-p)x} + \cdots + \binom{2n}{2n-1} e^{-2(n-1)x} + \binom{2n}{2n} e^{-2nx}$$

The first and last term, the second and second to last, etc., were taken, then it was generalized by obtaining the p^{th} term and the $(2n-p)^{th}$ term (Burton, 2007).

Making use of the fact that $\binom{2n}{p} = \binom{2n}{2n-p}$, $p = 0, \cdots, 2n$ (Burton, 2007; Guichar, 2017), it can be rewritten as:

$$\binom{2n}{p} e^{2(n-p)x} + \binom{2n}{2n-p} e^{-2(n-p)x} = \binom{2n}{p}\left[e^{2(n-p)x} + e^{-2(n-p)x}\right] = 2\binom{2n}{p} \cosh 2(n-p)x$$

With the substitution $p = n - k$, it was clearly see that (Swokowski, 1979):

$$\binom{2n}{p} \cosh 2(n-p)x = \binom{2n}{n-k} \cosh 2kx,$$

ioThis expression holds for every value of p and so it trivially holds for every k, since $p = n-k$.

Thus $(e^x + e^{-x})^{2n} = \binom{2n}{n} + 2\sum_{k=1}^{n}\binom{2n}{n-k}\cosh 2kx$

By simplifying, we obtain:

$$\sum_{k=1}^{n}\binom{2n}{n-k}\cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} - \frac{1}{2}\binom{2n}{n}$$

By taking into consideration the case $k=0$, the result follows:

$$\sum_{k=0}^{n}\binom{2n}{n-k}\cosh 2kx = \frac{1}{2}(e^x + e^{-x})^{2n} + \frac{1}{2}\binom{2n}{n}$$

Then attention given to finding an expression for $\phi(x) = \sum_{r=0}^{n}\binom{2n}{n-r}\sinh 2rx$ Clearly $\sum_{r=0}^{n}\binom{2n}{n-r}\sinh 2kx = \sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx$ since $\sinh(0) = 0$
So it was observed that:

$$\phi(x) = \sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx$$

An important formula is (Burton, 2007):

$$\binom{\alpha+1}{\beta+1} = \frac{(\alpha+1)!}{(\alpha-\beta)!(\beta+1)!} = \frac{(\alpha+1)\alpha!}{(\alpha-\beta)!(\beta+1)(\beta)!} = \frac{\alpha+1}{\beta+1}\frac{(\alpha)!}{(\alpha-\beta)!(\beta)!} = \frac{\alpha+1}{\beta+1}\binom{\alpha}{\beta}$$

That is $\binom{\alpha+1}{\beta+1} = \frac{\alpha+1}{\beta+1}\binom{\alpha}{\beta}$

Applying this formula, another relationship that was later found helpful was arrived at (Swokowski, 1979).

This helpful relationship is $\frac{n-r+1}{2n+1}\binom{2n+1}{n-r+1} = \binom{2n}{n-r}$

Thus $\phi(x) = \sum_{r=1}^{n}\frac{n-r+1}{2n+1}\binom{2n+1}{n-r+1}\sinh 2rx \quad \Rightarrow$
(see Swokowski, 1979)

Another result that was found useful is:

$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \binom{\alpha+1}{\beta+1}$ (This is known as Pascal's Identity (Burton, 2007):

Proof

$$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \frac{(\alpha)!}{(\alpha-\beta)!(\beta)!} + \frac{(\alpha)!}{(\alpha-\beta-1)!(\beta+1)!}$$

$$= \frac{(\alpha)!}{(\alpha-\beta)(\alpha-\beta-1)!(\beta)!} + \frac{(\alpha)!}{(\alpha-\beta-1)!(\beta+1)(\beta)!}$$
$$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{1}{(\alpha-\beta)} + \frac{1}{(\beta+1)}\right]$$
$$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{(\beta+1)+(\alpha-\beta)}{(\alpha-\beta)(\beta+1)}\right]$$
$$= \left[\frac{(\alpha)!}{(\alpha-\beta-1)!(\beta)!}\right]\left[\frac{1+\alpha}{(\alpha-\beta)(\beta+1)}\right]$$
$$= \frac{(\alpha+1)(\alpha)!}{(\alpha-\beta)(\alpha-\beta-1)!(\beta+1)(\beta)!}$$
$$= \frac{(\alpha+1)!}{(\alpha-\beta)!(\beta+1)!}$$
$$= \binom{\alpha+1}{\beta+1}$$

This clearly makes the relationship $\binom{2n+1}{n-r+1} = \binom{2n}{n-r} + \binom{2n}{n-r+1}$ valid. By substituting this into the present equation, we arrive at:

$$(2n+1)\phi(x) = (n+1)\sum_{r=1}^{n}\left[\binom{2n}{n-r} + \binom{2n}{n-r+1}\right]\sinh 2rx - \sum_{r=1}^{n} r\left[\binom{2n}{n-r} + \binom{2n}{n-r+1}\right]\sinh 2rx$$

$$= (n+1)\sum_{r=1}^{n}\binom{2n}{n-r}\sinh 2rx + (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r+1}\sinh 2rx$$

$$= (n+1)\phi(x) + (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r+1}\sinh 2rx$$

Thus

$$n\phi(x) = (n+1)\sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r}\sinh 2rx - \sum_{r=1}^{n} r\binom{2n}{n-r+1}\sinh 2rx \quad (11)$$

Let $Y_A = \sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx$

$$Y_B = \sum_{r=1}^{n} r\binom{2n}{n-r}\sinh 2rx$$

$$Y_C = \sum_{r=1}^{n} r\binom{2n}{n-r}\sinh 2rx$$

$$\Rightarrow n\phi(x) = (n+1)Y_A - Y_B - Y_C \quad (12)$$

Now $Y_A = \sum_{r=1}^{n}\binom{2n}{n-r+1}\sinh 2rx$ (13)

Let us make the substitution $r = p+1$ (Wrede and Speigel, 2010):

Thus $Y_A = \sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2(p+1)x$

$$Y_A = \sum_{p=0}^{n-1}\binom{2n}{n-p}\left[\sinh 2px\cosh 2x + \cosh 2px\sinh 2x\right]$$

$$= \left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2px\right]\cosh 2x + \left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px\right]\sinh 2x$$

$$= \left[\phi(x) - \sinh 2nx\right]\cosh 2x + \left[J(x) - \cosh 2nx\right]\sinh 2x$$

$$= \cosh 2x\phi(x) - \cosh 2x\sinh 2nx + \sinh 2xJ(x) - \sinh 2x\cosh 2nx$$

$$Y_A = \cosh 2x\phi(x) + \sinh 2xJ(x) - \sinh 2(n+1)x \quad (14)$$

$$Y_B = \sum_{r-1}^{n} r\binom{2n}{n-r}\sinh 2rx \quad (15)$$

$$= \frac{1}{2}\frac{d}{dx}\sum_{r=1}^{n}\binom{2n}{n-r}\cosh 2rx$$

$$= \frac{1}{2}\frac{d}{dx}\left[J(x) - \binom{2n}{n}\right]$$

$$Y_B = \frac{1}{2}J'(x) \quad (16)$$

$$Y_C = \sum_{r=1}^{n} r\binom{2n}{n-r+1}\sinh 2rx \quad (17)$$

$$= \frac{1}{2}\frac{d}{dx}\sum_{r=1}^{n}\binom{2n}{n-r+1}\cosh 2rx$$

With the substitution $r = p+1$, we arrive at

$$Y_C = \frac{1}{2}\frac{d}{dx}\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2(p+1)x$$

$$= \frac{1}{2}\frac{d}{dx}\left[\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px\cosh 2x + \sinh 2px\sinh 2x\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\cosh 2x\sum_{p=0}^{n-1}\binom{2n}{n-p}\cosh 2px + \sinh 2x\sum_{p=0}^{n-1}\binom{2n}{n-p}\sinh 2px\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\left(J(x) - \cosh 2nx\right)\cosh 2x + \left(\phi(x) - \sinh 2nx\right)\sinh 2x\right]$$

$$= \frac{1}{2}\begin{bmatrix}\left(J'(x) - 2n\sinh 2nx\right)\cosh 2x + \left(J(x) - \cosh 2nx\right)2\sinh 2x \\ + \left(\phi'(x) - 2n\cosh 2nx\right)\sinh 2x + \left(\phi(x) - \sinh 2nx\right)2\cosh 2x\end{bmatrix}$$

$$= \frac{1}{2}J'(x)\cosh 2x - (n+1)[\sinh 2nx\cosh 2x + \cosh 2nx\sinh 2x] + J(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2x + \phi(x)\cosh 2x$$

$$Y_C = \frac{1}{2}J'(x)\cosh 2x - (n+1)\sinh 2(n+1)x + J(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2x + \phi(x)\cosh 2x \quad (18)$$

It is clear that

$$n\phi(x) = (n+1)Y_A - Y_B - Y_C$$

So

$$n\phi(x) = \begin{pmatrix}(n+1)\cosh 2x\phi(x) + (n+1)\sinh 2xJ(x) - (n+n)\sinh 2(n+1)x - \frac{1}{2}J'(x) \\ -\left[\frac{1}{2}J'(x)\cosh 2x + j(x)\sinh 2x + \frac{1}{2}\phi'(x)\sinh 2n + \phi(x)\cosh 2x - (n+1)\sinh 2(n+1)x\right]\end{pmatrix}$$

Some of the terms cancel out and we are left with:

$$n\phi(x) = n\cosh 2x\phi(x) + n\sinh 2xJ(x) - \frac{1}{2}J'(x)\left(1 + \cosh 2x\right) - \frac{1}{2}\phi'(x)\sinh 2x \quad (19)$$

Digress (Swokowski, 1979; Wrede and Speigel, 2010):

$$J(x)=\sum_{r=0}^{n}\binom{2n}{n-r}\cosh 2rx=\frac{1}{2}\left[\left(e^{x}+e^{-x}\right)^{2n}+\binom{2n}{n}\right]$$

$$J'(x)=n\left(e^{x}+e^{-x}\right)^{2n-1}\left(e^{x}-e^{-x}\right)$$

$$1+\cosh 2x=2\cosh^{2}x$$

$$n\sinh 2xJ(x)-\frac{1}{2}J'(x)[1+\cosh 2x]=n\cosh x\sinh x\left[\left(e^{x}+e^{-x}\right)^{2n}+\binom{2n}{n}\right]-n\left(e^{x}+e^{-x}\right)^{2n-1}\left(e^{x}-e^{-x}\right)\cosh^{2}x$$

$$=n\cosh x\sinh x\left[\left(e^{x}+e^{-x}\right)^{2n}+\binom{2n}{n}\right]-n\cosh x\sinh x\left(e^{x}+e^{-x}\right)^{2n}$$

$$=n\cosh x\sinh x\binom{2n}{n}$$

$$=\frac{1}{2}n\sinh 2x\binom{2n}{n}$$

Making the above substitutions gives us

$$n\phi(x)=n\cosh 2x\phi(x)-\frac{1}{2}\phi'(x)\sinh 2x+\frac{1}{2}n\sinh 2x\binom{2n}{n}$$

$$\Rightarrow$$

$$\frac{1}{2}\phi'(x)\sinh 2x+n\phi(x)[1-\cosh 2x]=\frac{1}{2}n\sinh 2x\binom{2n}{n}$$

$$\Rightarrow$$

$$\phi'(x)\sinh x\cosh x-2n\phi(x)\sinh^{2}x=n\sinh x\cosh x\binom{2n}{n}$$

$$\Rightarrow\quad \phi'(x)\cosh x-2n\phi(x)\sinh x=n\cosh x\binom{2n}{n}$$

Thus

$$\phi'(x)-2n\phi(x)\tanh x=n\binom{2n}{n} \tag{20}$$

This first order differential equation can be solved by simply finding an integrating factor. Let (Bronson and Bredensteine, 2003):

$$P=-2n\tanh x$$

where R is the integrating factor, we use the known formula $R=e^{\int Pdx}$

Now $\int Pdx=\int -2n\tanh x dx=-2n\ln\cosh x$

So that R becomes $R=e^{-2n\ln\cosh x}=\left[e^{\ln\cosh x}\right]^{(-2n)}=\left(\cosh x\right)^{-2n}=\left(\operatorname{sech}^{2}x\right)^{n}$

The solution thus becomes $\frac{d}{dx}\left[\phi(x)\left(\operatorname{sech}^{2}x\right)^{n}=n\binom{2n}{n}\left(\operatorname{sech}^{2}x\right)^{n}\right]$

Thus

$$\phi(x)=n\binom{2n}{n}\left(\cosh^{2}x\right)^{n}\int_{0}^{x}\left(\operatorname{sech}^{2}t\right)^{n}dt \tag{21}$$

where t is a dummy variable and the lower limit of integration is 0 because $\phi(0)=0$.

THE NEW FORMULA

Going back to Equation 9,

$$H_{1}(x)=J(x)+\phi(x)$$

$$H_{1}(x)=\frac{1}{2}(e^{x}+e^{-x})^{2n}+\frac{1}{2}\binom{2n}{n}+n\binom{2n}{n}\left(\cosh^{2}x\right)^{n}\int_{0}^{x}\left(\operatorname{sech}^{2}t\right)^{n}dt \tag{22}$$

Making the substitution $x=\frac{1}{2}\ln\left(\frac{b}{ay}\right)$

It is seen that $e^{x}+e^{-x}=\sqrt{\frac{b}{ay}}+\sqrt{\frac{ay}{b}}$

So

$$h_{1}f(y)=\frac{1}{2}\left(\frac{b+ay}{\sqrt{aby}}\right)^{2n}+\frac{1}{2}\binom{2n}{n}+n\binom{2n}{n}\left(\frac{ay+b}{2\sqrt{aby}}\right)^{2n}\int_{0}^{x}\left(\operatorname{sech}^{2}t\right)^{n}dt \tag{23}$$

Thus the halfway summation

$$hf(y)=(ay)^{n}b^{n}\left[h_{1}(y)\right] \tag{24}$$

$$hf(y)=(ay)^{n}b^{n}\left[\frac{1}{2}\left(\frac{b+ay}{\sqrt{aby}}\right)^{2n}+\frac{1}{2}\binom{2n}{n}+n\binom{2n}{n}\left(\frac{ay+b}{2\sqrt{aby}}\right)^{2n}\int_{0}^{x}\left(\operatorname{sech}^{2}t\right)^{n}dt\right] \tag{25}$$

SIMPLE ILLUSTRATION TO TEST RESULT

Let us test the result for a small value of n because we can easily cope with it by hand. Say for instance $n=2$, the present formula simplifies to

$$hf(y)=(ay)^{2}b^{2}\left[\frac{1}{2}\left(\frac{ay+b}{\sqrt{aby}}\right)^{4}+\frac{1}{2}\binom{4}{2}+2\binom{4}{2}\left(\frac{ay+b}{2\sqrt{aby}}\right)^{4}\int_{0}^{x}\left(\operatorname{sech}^{2}t\right)^{2}dt\right]$$

$$= (ay)^2 b^2 \left[\frac{1}{2} \frac{(ay+b)^4}{(aby)^2} + \frac{1}{2}(6) + 2(6) \frac{(ay+b)^4}{16(aby)^2} \int_0^x \left(\operatorname{sech}^2 t\right)^2 dt \right]$$

$$= (ay)^2 b^2 \left[\frac{1}{2} \frac{(ay+b)^4}{(aby)^2} + 3 + \frac{3}{4} \frac{(ay+b)^4}{(aby)^2} \int_0^x \left(\operatorname{sech}^2 t\right)^2 dt \right]$$

Consider $\int_0^x (\operatorname{sech}^2 t)dt = \int_0^x (\operatorname{sech}^2 t)(\operatorname{sech}^2 t)dt$

$$= \int_0^x \operatorname{sech}^2 t \, d(\tanh t)$$

$$= \int_0^x (1 - \tanh^2 t) \, d(\tanh t)$$

$$= \left[\tanh t - \frac{1}{3} \tanh^3 t \right]_0^x$$

$$= \tanh x - \frac{1}{3} \tanh^3 x$$

$$= \frac{\sqrt{\frac{b}{ay}} - \sqrt{\frac{ay}{b}}}{\sqrt{\frac{b}{ay}} + \sqrt{\frac{ay}{b}}} - \frac{1}{3} \left(\frac{\sqrt{\frac{b}{ay}} - \sqrt{\frac{ay}{b}}}{\sqrt{\frac{b}{ay}} + \sqrt{\frac{ay}{b}}} \right)^3$$

$$= \frac{b - ay}{b + ay} - \frac{1}{3} \left(\frac{b - ay}{b + ay} \right)^3$$

Thus, we have (Swokowski, 1979; Bronson and Bredensteiner, 2003):

$$hf(y) = (ay)^2 b^2 \left[\frac{1}{2} \frac{(ay+b)^4}{(aby)^2} + 3 + \frac{3}{4} \frac{(ay+b)^4}{(aby)^2} \left[\frac{b-ay}{b+ay} - \frac{1}{3} \left(\frac{b-ay}{b+ay} \right)^3 \right] \right]$$

$$hf(y) = \frac{1}{2}(ay+b)^4 + 3a^2b^2y^2 + \frac{3}{4}(ay+b)^4 \left[\frac{b-ay}{b+ay} - \frac{1}{3} \left(\frac{b-ay}{b+ay} \right)^3 \right]$$

$$hf(y) = \frac{1}{2}(ay+b)^4 + 3a^2b^2y^2 + \frac{3}{4}(ay+b)^3(b-ay) - \frac{1}{4}(b+ay)(b-ay)^3$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2b^2y^2 + \frac{1}{4}(b+ay)(b-ay)\left[3(b+ay)^2 - (b-ay)^2\right]$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2b^2y^2 + \frac{1}{4}\left(b^2 - (ay)^2\right)\left[2b^2 + 8aby + 2a^2y^2\right]$$

$$= \frac{1}{2}(ay+b)^4 + 3a^2b^2y^2 + \frac{1}{2}\left(b^2 - (ay)^2\right)\left(b^2 + 4aby + a^2y^2\right)$$

$$= \frac{1}{2}\left[(ay)^4 + 4(ay)^3 b + 6(ay)^2 b^2 + 4(ay)b^3 + b^4\right] + 3a^2b^2y^2 + \frac{1}{2}\left[b^4 + 4ab^3y + a^2b^2y^2 - a^2b^2y^2 - 4a^3by^3 - a^4y^4\right]$$

$$= \frac{1}{2}\left[6a^2b^2y^2 + 8ab^3y + 2b^4\right] + 3a^2b^2y^2$$

$$= 3a^2b^2y^2 + 4ab^3y + b^4 + 3a^2b^2y^2$$

$$= 6a^2b^2y^2 + 4ab^3y + b^4$$

This is the exact expression for the upper half way expansion of $(ay+b)^4$.

OTHER APPLICATIONS OF THE HALFWAY FORMULA

A series for $(1-q)^{-n}$ will be built. If we happen to take the painful task of extending the Pascal's triangle by inducing the terms underneath which turns out to be the lower half of another triangle, so that the combination of these two gives a square (Figure 1). Following the direction of the arrows, we see that, the first arrow depicts the coefficient of the first few terms of the expansion of the series $(1-q)^{-1}$, while the second gives the first few terms of the coefficients of the expansion of $(1-q)^{-2}$, etc. From Figure 1, a series is built (Kalman, 1993):

$(1-q)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} q^k \quad n \geq 1$ (n is a natural number) $|q| < 1$ (26)

From Equation 26, we obtain the series:

$$\left(1 - \tfrac{1}{q}\right)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} q^{-k} \quad |q| > 1. \qquad (27)$$

Furthermore, from Equation 27, if we take the sum of the first $n+1$ terms of the series and then multiply this series by q^n, a series will be obtained whose first $n+1$ coefficients are the same as those in Equation 26, but with the order of its powers reversed (in descending order). The new series described so far can be expressed by the formula:

$$\varphi\left[q^n (1 - \tfrac{1}{q})^{-n}\right] = \sum_{k=0}^{n} \binom{n-1+k}{n-1} q^{n-k} \qquad (28)$$

Now in other to see the usefulness of Equation 28, let us extract from Equation 1, the expression:

$$\sum_{k=0}^{n} \binom{2n}{n-k} \left(\frac{y-\sigma}{\sigma} \right)^k = \sum_{k=0}^{n} \binom{2n}{n-k} \left(\frac{y}{\sigma} - 1 \right)^k$$

Substituting this in Equation 28, gives us:

$$\sum_{k=0}^{n} \binom{2n}{n-k} \left(\frac{y}{\sigma} - 1 \right)^k = \sum_{k=0}^{n} \binom{n-1+k}{n-1} \left(\frac{y}{\sigma} \right)^{n-k} \qquad (29)$$

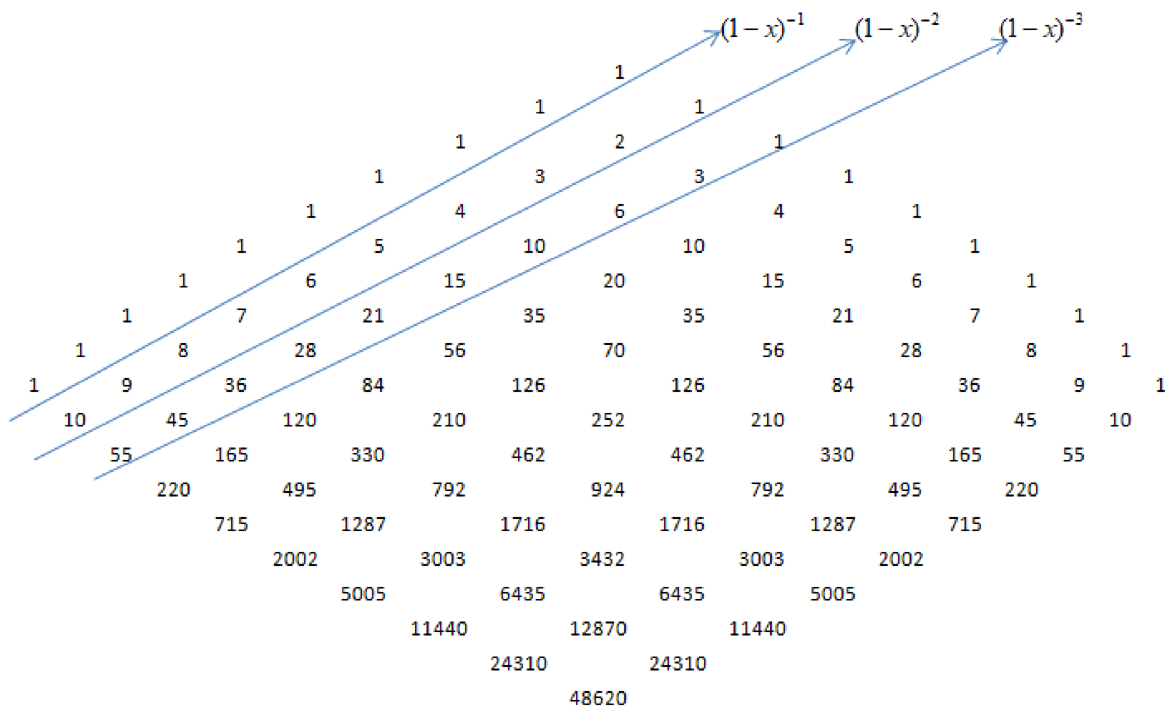

Figure 1. The induced Pascal's square.

A proof of Equation 29 is yet to be obtained; but from simple illustrations it is true. Its accuracy for small values of n has been tested. Well say for instance $n=3$ and $q=\frac{y}{\sigma}$

Right hand side will be (Guichard, 2017):

$$\sum_{k=0}^{3}\binom{2+k}{2}q^{3-k}=\binom{2}{2}q^3+\binom{3}{2}q^2+\binom{4}{2}q+\binom{5}{2}=q^3+3q^2+6q+10$$

For the left hand side (Burton, 2007), we have:

$$\sum_{k=0}^{3}\binom{6}{3-k}(q-1)^k=\binom{6}{3}+\binom{6}{2}(q-1)+\binom{6}{1}(q-1)^2+\binom{6}{0}(q-1)^3$$
$$=20+15(q-1)+6(q^2-2q+1)+(q^3-3q^2+3q-1)$$
$$=(20-15+6-1)+(15-12+3)q+(6-3)q^2+q^3$$
$$=10+6q+3q^2+q^3$$

So left hand side = right hand side.

From Equation 29, if the substitution $q=\frac{y}{\sigma}$ is made, then we obtain:

$$\sum_{k=0}^{n}\binom{2n}{n-k}(q-1)^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(q)^{n-k}$$

When $q-1=s$ is substituted, the equation reduces to:

$$\sum_{k=0}^{n}\binom{2n}{n-k}s^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(s+1)^{n-k} \tag{30}$$

This new Equation 30 is clearly the halfway series expansion of $(s+1)^{2n}$, that is

$$h(s+1)^{2n}=\sum_{k=0}^{n}\binom{2n}{n-k}s^k=\sum_{k=0}^{n}\binom{n-1+k}{n-1}(s+1)^{n-k} \tag{31}$$

A RELATIONSHIP DERIVED FROM THE INDUCED PASCAL'S SQUARE

The formula

$$\sum_{k=0}^{m}\binom{n-1+k}{n-1}=\binom{n+m}{n} \tag{32}$$

This is valid for all $m\geq 1$, also may be found useful (Enochs, 2004) and is also deducible from Figure 1.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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