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Construction of Stable High Order One-Block Methods Using Multi-Block Triple

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Authors' contributions

This work was carried out in collaboration among all authors. Authors IJA and MOD designed the study and wrote the first draft of the manuscript. Authors IJA and KU managed the analyses of the study. Author IJA managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This paper deals with the construction of l-stable implicit one-block methods for the solution of stiff initial value problems. The constructions are done using three different multi-block methods. The first multi-block method is composed using Generalized Backward Differentiation Formula (GBDF) and Backward Differentiation Formula (BDF), the second is composed using Reversed Generalized Adams Moulton (RGAM) and Generalized Adams Moulton (GAM) while the third is composed using Reversed Adams Moulton (RAM) and Adams Moulton (AM). Shift operator is then applied to the combination of the three multi-block methods in such a manner that the resultant block is a one-block method and self-starting. These one-block methods are $l-stable$ up to order six and $l(\alpha)-stable$ with $\alpha = 79.75^o$ at order ten. Numerical experiments show that they are good for solving stiff initial problems.

Keywords: l-stable; multi-block; stiff initial value problem; one-block and self-starting.

Subject Classification: 65L04; 65L05; 65L06.

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1 Introduction

Consider the problem of finding the numerical solution $y(t)$ to the stiff initial value problems (ivp) in ordinary differential equations (ode) of the form

$$\begin{aligned} & y'(t) = f(t, y(t)); \quad y(t_0) = y_0; \quad t \in [a, b]; \\ & f : \Re \times \Re^m \to \Re^m; \quad y : \Re \to \Re^m \end{aligned} \tag{1.1}$$

Stiff problems can only be handled without so much step size restrictions by A-stable methods but these methods are difficult to come by because of Dahlquist order barrier theorem [1]. In [2], it was pointed out that many researchers have circumvented this order barrier and constructed high order, A-stable methods through unconventional means. In recent literatures, the focus is on block methods as a means of circumventing this barrier theorem. These methods are composed using different linear multistep formulas (LMF), (see [3,4,5,6,7,8,9]. In this paper, we show how one-block methods can be constructed using different multi-block methods as opposed to the known convention of using different single LMF. This paper is divided into five sections; section 2 started with a brief review of LMF and describes how to construct the family of methods. Section 3 gives the stability analysis of the methods, section 4 deals with numerical experiments while the conclusion is given in section 5.

2 Construction of the Methods

The classical linear multistep formula which is given by

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h_n \sum_{r=0}^{k} \beta_r f(t_{n+r}, y_{n+r}) \tag{2.1}$$

where the step number $k > 1$ and $h_n = t_{n+1} - t_n$ is a variable step length, $\{\alpha_r\}_{r=0}^{k}$ and $\{\beta_r\}_{r=0}^{k}$ are real constants and both are not zero. If (2.1) is applied to the scalar test equation

$$y' = \lambda y, \ \mathrm{Re}(\lambda) < 0. \tag{2.2}$$

it yields the stability polynomial

$$\rho(z) - h\sigma(z) = 0 \tag{2.3}$$

where

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r \tag{2.4}$$

Now let us redefine (2.4) as

$$\rho(z) = \sum_{r=0}^{k} A_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} B_r z^r \tag{2.5}$$

where $\{A_i\}_{i=0}^{k}$ and , $\{B_i\}_{i=0}^{k}$ are matrices (block coefficients), then (2.1) becomes a linear multi-block methods (LMBM)

$$\sum_{r=0}^{k} A_r y_{n+r} = h_n \sum_{r=0}^{k} B_r f(t_{n+r}, y_{n+r}) \tag{2.6}$$

Equation (2.6) is what is needed in the construction of the new block methods instead of the usual (2.1). The methodology for the construction is explained in the following proposition:

Theorem

Let the family of Linear Multi-Block Methods (LMBM) $\left\{\rho_k^{[j]}(R), \sigma_k^{[j]}(R)\right\}_{j=1,k=1}^{m,T}$ *be given, that is,*

$$\rho_k^{[j]}(E)Y_n = h\,\sigma_k^{[j]}(E)F_n\,;\; j = 1(1)m,\;\; k = 1(1)T \tag{2.7}$$

with $\left\{\rho_k^{[j]}, \sigma_k^{[j]}\right\}$ *for a fixed j forming a family of variable order* $p_{k,j}$ *of variable step number k. Then the resultant system of composite LMBM*

$$E^i \rho_k^{[j]}(E)Y_n = hE^i \sigma_k^{[j]}(E)F_n\;;\; i = 0(1)k - l\;\;;\; j = 1,2,\ldots,m \text{ (the number of LMBM)} \tag{2.8}$$

arising from the E-operator transformation of (2.7) *can be composed as the one-block method*

$$C_1 Y_{n+1} + C_0 Y_n = h(D_1 F_{n+1} + D_0 F_n)\,;\; \det(C_1) \neq 0 \tag{2.9}$$

if k is chosen such that l is an integer given as

$$l = \frac{k(ms - 1 - s) + ms}{s(m-1)}\;;\;\; k \geq 4; m, s \geq 2; (s \text{ is the number of rows in each LMBM})$$
$$\text{and } k - l \geq 0. \tag{2.10}$$

where Y_{n+1}, Y_n; F_{n+1} *and* F_n $n = 0,1,2,\ldots$ *are as defined below and* C_1, C_0, D_1, D_0 *are square matrices also defined below for a fixed s and m.*

$$C_0 = \left(\begin{array}{c|c} & A_0^{[1]} \\ & A_0^{[2]} \\ & \cdot \\ & \cdot \\ & \cdot \\ & A_0^{[m]} \\ O & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{array}\right)_{(k+s(k-l))\times(k+s(k-l))} ; \quad D_0 = \left(\begin{array}{c|c} & B_0^{[1]} \\ & B_0^{[2]} \\ & \cdot \\ & \cdot \\ & \cdot \\ & B_0^{[m]} \\ O & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{array}\right)_{(k+s(k-l))\times(k+s(k-l))} \qquad (2.11)$$

$$C_1 = \begin{pmatrix}
A_1^{[1]} & \cdot & \cdot & \cdot & A_k^{[1]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & 0 \\
A_1^{[m]} & \cdot & \cdot & \cdot & A_k^{[m]} & 0 & \cdot & \cdot & & & & & & & & 0 \\
A_0^{[1]} & A_1^{[1]} & \cdot & \cdot & \cdot & A_k^{[1]} & \cdot & \cdot & & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & & \cdot & 0 & & & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & & & & \cdot \\
A_0^{[m]} & A_1^{[m]} & & & & A_k^{[m]} & 0 & & & & & & & & & \cdot \\
0 & A_0^{[1]} & A_1^{[1]} & & & & A_k^{[1]} & & & & & & & & & \cdot \\
0 & \cdot & \cdot & & & \cdot & \cdot & & & & & & & & & \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & \\
& A_0^{[m]} & A_1^{[m]} & \cdot & \cdot & \cdot & A_k^{[m]} & 0 & & & & & & & & \\
\cdot & \cdot & \cdot & & & & \cdot & & & & & & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & & & \cdot & & & \cdot & \cdot & & & & & 0 \\
\cdot & & & & & & & & & \cdot & A_0^{[1]} & & & & & A_k^{[1]} \\
& & & & & & & & & \cdot & \cdot & & & & & \cdot \\
0 & \cdot & \cdot & \cdot & & & & & & 0 & A_0^{[m]} & & \cdot & \cdot & \cdot & A_k^{[m]}
\end{pmatrix}_{(k+s(k-l))\times(k+s(k-l))}$$

$$
D_1 = \begin{pmatrix}
B_1^{[1]} & . & . & . & B_k^{[1]} & 0 & 0 & 0 & . & . & . & & & & 0 \\
. & & & & & & & & & & & & & & \\
. & . & . & . & . & . & . & . & & & & & & & 0 \\
. & & & & & & & & & & & & & & \\
B_1^{[m]} & . & . & . & B_k^{[m]} & 0 & . & . & & & & & & & 0 \\
B_0^{[1]} & B_1^{[1]} & . & . & . & B_k^{[1]} & . & . & & & & & & & 0 \\
. & . & . & . & & . & 0 & & & & & & & & 0 \\
. & . & & & & . & & & & & & & & & \\
. & . & . & . & . & . & 0 & & & & & & & & . \\
B_0^{[m]} & B_1^{[m]} & & & & B_k^{[m]} & 0 & & & & & & & & . \\
0 & B_0^{[1]} & B_1^{[1]} & & & B_{k-1}^{1} & B_k^{[1]} & & & & & & & & . \\
0 & . & . & & & . & . & & & & & & & & \\
 & . & . & . & . & . & . & & & & & & & & \\
 & B_0^{[m]} & B_1^{[m]} & . & . & B_{k-1}^{[m]} & B_k^{[m]} & 0 & & & & & & & \\
 & . & . & & & & . & & & & & & & & \\
. & . & . & & & & . & & & & & . & . & . & \\
. & . & . & & & & . & & & . & . & & & & 0 \\
. & & & & & & & & & . & B_0^{[1]} & & & & B_k^{[1]} \\
 & & & & & & & & & . & . & & & & . \\
0 & . & . & . & & & & & & 0 & B_0^{[m]} & & . & . & . \; B_k^{[m]}
\end{pmatrix}_{(k+s(k-l))\times(k+s(k-l))}
$$

$$
Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+k+s(k-l)})^T \,; \quad Y_n = (y_{n-(k+s(k-l))+1}, y_{n-2k+l+2}, \ldots, y_{n-1}, y_n)^T \,; \quad (2.12)
$$

$$
F_{n+1} = (f_{n+1}, f_{n+2}, \ldots, f_{n+k+s(k-l)})^T \,; \quad F_n = (f_{n-(k+s(k-l))+1}, f_{n-2k+l+2}, \ldots, f_{n-1}, f_n)^T
$$

$n = 0,1,2,\ldots$

Proof:

Notice that the *E*-operator is effectively applied *k-l* times on the system of LMBM $\{\rho_k^{[j]},\ \sigma_k^{[j]}\}_{k,j}$. Thus there are $(k + s(k-l)) \times (k + s(k-l))$ unknown solution points captured in the block of solution $Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+k+s(k-l)})^T$ .By this block definition in (2.9) is realized if the coefficient matrices C_1, C_0, D_1, D_0 are square matrices of dimension $(k + s(k-l)) \times (k + s(k-l))$.

This simply imply that $ms + ms(k-l) = k + s(k-l)$ so that *l* is as in (2.10) and for a fixed *m* and *s*, *k* is chosen such that $k - l \geq 0$. ∎

In particular:

(1.) $m = 2$; $s = 2$; $l = \dfrac{k+4}{2}$; $k = 4, 6, 8, 10, \ldots$

(2.) $m = 3\ ; s = 2;\ l = \frac{3k+6}{4}\ ;\ k = 6,10,14,...$

When $k - l = 0$, the method requires zero shifting. This is so if $ms=k$. However, the case of interest in this paper is when m = 3 and s = 2.Consider the family composed using GBDF/BDF [5], RGAM/GAM and RAM/AM [4] methods, the coefficients are respectively given below:

The method constructed using the pair of GBDF and BDF of order 6, that is k=6.

$$\begin{pmatrix} \frac{2}{5} & \frac{-1}{30} \\ -6 & \frac{49}{20} \end{pmatrix} Y_{n+3} + \begin{pmatrix} \frac{-4}{3} & \frac{7}{12} \\ \frac{-20}{3} & \frac{15}{2} \end{pmatrix} Y_{n+2} + \begin{pmatrix} \frac{-2}{15} & \frac{1}{2} \\ \frac{-6}{5} & \frac{15}{4} \end{pmatrix} Y_{n+1} + \begin{pmatrix} 0 & \frac{1}{60} \\ 0 & \frac{1}{6} \end{pmatrix} Y_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} F_{n+3} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_{n+2} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} F_{n+1} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} F_n \tag{2.13}$$

The method constructed using the pair of RGAM and GAM of order 7, that is k=6.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} Y_{n+3} + \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} Y_{n+2} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} Y_{n+1} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} Y_n = \begin{pmatrix} \frac{67}{2520} & \frac{-191}{60480} \\ \frac{23}{504} & \frac{-271}{60480} \end{pmatrix} F_{n+3} + \begin{pmatrix} \frac{586}{945} & \frac{-2257}{20160} \\ \frac{-586}{945} & \frac{-10273}{20160} \end{pmatrix} F_{n+2} + \begin{pmatrix} \frac{-23}{504} & \frac{10273}{20160} \\ \frac{-67}{2520} & \frac{2257}{20160} \end{pmatrix} F_{n+1} + \begin{pmatrix} 0 & \frac{271}{60480} \\ 0 & \frac{191}{60480} \end{pmatrix} F_n \tag{2.14}$$

The method constructed using the pair of RAM and AM of order 7, that is k=6.

$$\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} Y_{n+3} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} Y_{n+2} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y_{n+1} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} Y_n = \begin{pmatrix} \frac{263}{2520} & \frac{-863}{60480} \\ \frac{2713}{2520} & \frac{19087}{60480} \end{pmatrix} F_{n+3} + \begin{pmatrix} \frac{586}{945} & \frac{-6737}{20160} \\ \frac{586}{945} & \frac{-15487}{20160} \end{pmatrix} F_{n+2} + \begin{pmatrix} \frac{2713}{2520} & \frac{-15487}{20160} \\ \frac{263}{2520} & \frac{-6737}{20160} \end{pmatrix} F_{n+1} + \begin{pmatrix} 0 & \frac{19087}{60480} \\ 0 & \frac{-863}{60480} \end{pmatrix} F_n \tag{2.15}$$

The three multi-block (Three-block) methods are then used to construct a one-block method given as in (2.9)where

$$C_1 = \begin{pmatrix} \frac{-2}{15} & \frac{1}{2} & \frac{-4}{3} & \frac{7}{12} & \frac{2}{5} & \frac{-1}{30} \\ \frac{-6}{5} & \frac{15}{4} & \frac{-20}{3} & \frac{15}{2} & -6 & \frac{49}{20} \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix};\ C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{60} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_0=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\frac{271}{60480}\\0&0&0&0&0&\frac{191}{60480}\\0&0&0&0&0&\frac{19087}{60480}\\0&0&0&0&0&\frac{-863}{60480}\end{pmatrix};\quad D_1=\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&1\\\frac{-23}{504}&\frac{10273}{20160}&\frac{586}{945}&\frac{-2257}{20160}&\frac{67}{2520}&\frac{-191}{60480}\\\frac{-67}{2520}&\frac{2257}{20160}&\frac{-586}{945}&\frac{-10273}{20160}&\frac{23}{504}&\frac{-271}{60480}\\\frac{2713}{2520}&\frac{-15487}{20160}&\frac{586}{945}&\frac{-6737}{20160}&\frac{263}{2520}&\frac{-863}{60480}\\\frac{263}{2520}&\frac{-6737}{20160}&\frac{586}{945}&\frac{-15487}{20160}&\frac{2713}{2520}&\frac{19087}{60480}\end{pmatrix};$$

Case of k=10, five-block methods constructed are

$$\begin{pmatrix}\frac{1}{63}&\frac{-1}{840}\\-10&\frac{7381}{2520}\end{pmatrix}Y_{n+5}+\begin{pmatrix}\frac{4}{7}&\frac{-3}{28}\\-40&\frac{45}{2}\end{pmatrix}Y_{n+4}+\begin{pmatrix}\frac{-6}{5}&\frac{11}{30}\\\frac{-252}{5}&\frac{105}{2}\end{pmatrix}Y_{n+3}+\begin{pmatrix}\frac{-4}{21}&\frac{1}{2}\\\frac{-120}{7}&35\end{pmatrix}Y_{n+2}+\begin{pmatrix}\frac{-1}{105}&\frac{3}{56}\\\frac{-10}{9}&\frac{45}{8}\end{pmatrix}Y_{n+1}+ \tag{2.16}$$

$$\begin{pmatrix}0&\frac{1}{1260}\\0&\frac{1}{10}\end{pmatrix}Y_n=\begin{pmatrix}0&0\\0&1\end{pmatrix}F_{n+5}+\begin{pmatrix}0&0\\0&0\end{pmatrix}F_{n+4}+\begin{pmatrix}0&1\\0&0\end{pmatrix}F_{n+3}+\begin{pmatrix}0&0\\0&0\end{pmatrix}F_{n+2}+\begin{pmatrix}0&0\\0&0\end{pmatrix}F_{n+1}+\begin{pmatrix}0&0\\0&0\end{pmatrix}F_n$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+5}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+4}+\begin{pmatrix}1&0\\1&-1\end{pmatrix}Y_{n+3}+\begin{pmatrix}0&-1\\0&0\end{pmatrix}Y_{n+2}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+1}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_n=$$

$$\begin{pmatrix}\frac{32309}{17107200}&\frac{-14797}{95800320}\\\frac{292531}{119750400}&\frac{-90817}{479001600}\end{pmatrix}F_{n+5}+\begin{pmatrix}\frac{163459}{3991680}&\frac{-1746433}{159667200}\\\frac{1394959}{19958400}&\frac{-493837}{31933440}\end{pmatrix}F_{n+4}+\begin{pmatrix}\frac{379571}{623700}&\frac{-3216337}{26611200}\\\frac{-379571}{623700}&\frac{-14296081}{26611200}\end{pmatrix}F_{n+3}+$$

$$\begin{pmatrix}\frac{-1394959}{19958400}&\frac{14296081}{26611200}\\\frac{-163459}{3991680}&\frac{3216337}{26611200}\end{pmatrix}F_{n+2}+\begin{pmatrix}\frac{-292531}{119750400}&\frac{493837}{31933440}\\\frac{-32309}{17107200}&\frac{1746433}{159667200}\end{pmatrix}F_{n+1}+\begin{pmatrix}0&\frac{90817}{479001600}\\0&\frac{14797}{95800320}\end{pmatrix}F_n \tag{2.17}$$

$$\begin{pmatrix}0&0\\-1&1\end{pmatrix}Y_{n+5}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+4}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+3}+\begin{pmatrix}0&0\\0&0\end{pmatrix}Y_{n+2}+\begin{pmatrix}1&0\\0&0\end{pmatrix}Y_{n+1}+\begin{pmatrix}0&-1\\0&0\end{pmatrix}Y_n=$$

$$\begin{pmatrix}\frac{9071219}{119750400}&\frac{-3250433}{479001600}\\\frac{164046413}{119750400}&\frac{26842253}{95800320}\end{pmatrix}F_{n+5}+\begin{pmatrix}\frac{23643791}{19958400}&\frac{-12318413}{31933440}\\\frac{12051709}{3991680}&\frac{-296725183}{159667200}\end{pmatrix}F_{n+4}+\begin{pmatrix}\frac{2227571}{623700}&\frac{-21677723}{8870400}\\\frac{2227571}{623700}&\frac{-33765029}{8870400}\end{pmatrix}F_{n+3}+$$

$$\begin{pmatrix}\frac{12051709}{3991680}&\frac{-33765029}{8870400}\\\frac{23643791}{19958400}&\frac{-21677723}{8870400}\end{pmatrix}F_{n+2}+\begin{pmatrix}\frac{164046413}{119750400}&\frac{-296725183}{159667200}\\\frac{9071219}{119750400}&\frac{-12318413}{31933440}\end{pmatrix}F_{n+1}+\begin{pmatrix}0&\frac{26842253}{95800320}\\0&\frac{-3250433}{479001600}\end{pmatrix}F_n \tag{2.18}$$

Putting all the three multi-block methods together, we have (2.9) where

$$C_1 = \begin{pmatrix} \frac{-1}{105} & \frac{3}{56} & \frac{-4}{21} & \frac{1}{2} & \frac{-6}{5} & \frac{11}{30} & \frac{4}{7} & \frac{-3}{28} & \frac{1}{63} & \frac{-1}{840} & 0 & 0 \\ \frac{-10}{9} & \frac{45}{8} & \frac{-120}{7} & 35 & \frac{-252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{1}{1260} & \frac{-1}{105} & \frac{3}{56} & \frac{-4}{21} & \frac{1}{2} & \frac{-6}{5} & \frac{11}{30} & \frac{4}{7} & \frac{-3}{28} & \frac{1}{63} & \frac{-1}{840} \\ 0 & \frac{1}{10} & \frac{-10}{9} & \frac{45}{8} & \frac{-120}{7} & 35 & \frac{-252}{5} & \frac{105}{2} & -40 & \frac{45}{2} & -10 & \frac{7381}{2520} \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix};$$

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1260} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-292531}{119750400} & \frac{493837}{31933440} & \frac{-1394959}{19958400} & \frac{14296081}{26611200} & \frac{379571}{623700} & \frac{-3216337}{26611200} & \frac{163459}{3991680} & \frac{-1746433}{159667200} & \frac{32309}{17107200} & \frac{-14797}{95800320} & 0 & 0 \\ \frac{-32309}{17107200} & \frac{1746433}{159667200} & \frac{-163459}{3991680} & \frac{3216337}{26611200} & \frac{-379571}{623700} & \frac{-14296081}{26611200} & \frac{1394959}{19958400} & \frac{-493837}{31933440} & \frac{292531}{119750400} & \frac{-90817}{479001600} & 0 & 0 \\ \frac{164046413}{119750400} & \frac{-296725183}{159667200} & \frac{12051709}{3991680} & \frac{-33765029}{8870400} & \frac{2227571}{623700} & \frac{-21677723}{8870400} & \frac{23643791}{19958400} & \frac{-12318413}{31933440} & \frac{9071219}{119750400} & \frac{-3250433}{479001600} & 0 & 0 \\ \frac{9071219}{119750400} & \frac{-12318413}{31933440} & \frac{23643791}{19958400} & \frac{-21677723}{8870400} & \frac{2227571}{623700} & \frac{-33765029}{8870400} & \frac{12051709}{3991680} & \frac{-296725183}{159667200} & \frac{164046413}{119750400} & \frac{26842253}{95800320} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{90817}{479001600} & \frac{-292531}{119750400} & \frac{493837}{31933440} & \frac{-1394959}{19958400} & \frac{14296081}{26611200} & \frac{379571}{623700} & \frac{-3216337}{26611200} & \frac{163459}{3991680} & \frac{-1746433}{159667200} & \frac{32309}{17107200} & \frac{-14797}{95800320} \\ 0 & \frac{14797}{95800320} & \frac{-32309}{17107200} & \frac{1746433}{159667200} & \frac{-163459}{3991680} & \frac{3216337}{26611200} & \frac{-379571}{623700} & \frac{-14296081}{26611200} & \frac{1394959}{19958400} & \frac{-493837}{31933440} & \frac{292531}{119750400} & \frac{-90817}{479001600} \\ 0 & \frac{26842253}{95800320} & \frac{164046413}{119750400} & \frac{-296725183}{159667200} & \frac{12051709}{3991680} & \frac{-33765029}{8870400} & \frac{2227571}{623700} & \frac{-21677723}{8870400} & \frac{23643791}{19958400} & \frac{-12318413}{31933440} & \frac{9071219}{119750400} & \frac{-3250433}{479001600} \\ 0 & \frac{-3250433}{479001600} & \frac{9071219}{119750400} & \frac{-12318413}{31933440} & \frac{23643791}{19958400} & \frac{-21677723}{8870400} & \frac{2227571}{623700} & \frac{-33765029}{8870400} & \frac{12051709}{3991680} & \frac{-296725183}{159667200} & \frac{164046413}{119750400} & \frac{26842253}{95800320} \end{pmatrix}$$

$$D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90817}{479001600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14797}{95800320} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{26842253}{95800320} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3250433}{479001600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

3 Stability of the Implicit One-Block Methods

When (2.9) is applied to the test equation (2.2), it yields the characteristics equation.

$$\pi(w,z) = \det(C_1 w + C_0 - z(D_1 w + D_0)), \quad z = \lambda h \tag{3.1}$$

For order k = 6,

$$\pi(w,z) = -\frac{571w^5}{720} + \frac{571w^6}{720} - \frac{9647w^5 z}{5040} - \frac{2876w^6 z}{1008} - \frac{61333w^5 z^2}{30240} + \frac{145717w^6 z^2}{30240} - \frac{26147w^5 z^3}{21600} - \frac{764341w^6 z^3}{151200} - \frac{4517w^5 z^4}{10800} + \frac{270707w^6 z^4}{75600} - \frac{6109w^6 z^5}{90720} - \frac{110011w^6 z^5}{64800} + \frac{3223w^6 z^6}{7560}$$

The region of absolute stability R_A associated with (2.9) is the set

$$R_A = \{z \in \mathbb{C} : |w_j(z)| \le 1,\ j = 1(1)(k + s(k-l))\} \tag{3.2}$$

For order 6 above $w_j(z),\ j = 1(1)6$ are given below

$$\{\{w \to 0\}\{w \to 0\},\{w \to 0\},\{w \to 0\},\{w \to 0\},$$
$$\{w \to \frac{359730 + 868230z + 919995z^2 + 549087z^3 + 189714z^4 + 30545z^5}{359730 - 1290150z + 2185755z^2 - 2293023z^3 + 1624242z^4 - 770077z^5 + 193380z^6}\}\}$$

The only non-zero value of $w(z)$ for this family of methods are given as a rational function $T(z) = \dfrac{P(z)}{Q(z)}$. where $P(z)$ and $Q(z)$ are polynomials. From the above $k = 6$,

$$T(z)=\frac{359730+868230z+919995z^2+549087z^3+189714z^4+30545z^5}{359730-1290150z+2185755z^2-2293023z^3+1624242z^4-770077z^5+193380z^6}$$

This value tends to zero as z tends to infinity.

Definition 1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in C^+ (see [10]).

The roots of $Q(z)$ are

{{z→0.2210288675951737 -1.2587046977754033i},{z→0.2210288675951737 +1.2587046977754033i},{z→0.7560441897235561 - 0.701940199394596i}, {z→0.7560441897235561 +0.701940199394596i}, {z→1.0140247811340575 - 0.2047642418277674i},{z→1.0140247811340575 +0.2047642418277674i}}

They are all contained in C^+.

Definition 2: A one block method is *A*-stable if and only if it is stable on the imaginary axis (*I*-stable) [11]: That is $T(iy)\le 1$ for all $y\in\Re$, and *T*(z) is analytic for $z<0$ (i.e. $Q(z)$ does not have roots with negative or zero real parts), *I*-stability is equivalent to the fact that the Norsett polynomial defined by

$$G(y)=|Q(iy)|^2-|P(iy)|^2=Q(iy)Q(-iy)-P(iy)P(-iy) \tag{3.3}$$

satisfies $G(y)>0$ for all $y\in\Re$ [11].

Definition 3: A block method is said to be $L-Stable$ if it is $A-Stable$ and also $T(z)\to 0$ as $z\to\infty$ [12].

The none zero solution, $T(z)$ of order 6 has no pole in C^-, all the roots of $Q(z)$ are contained in C^+. The orders 6 satisfies condition (3.3) and definitions 1 and 2, therefore it is $L-Stable$.

Definition 4: A LMF is said to be A(α)-Stable, with $\alpha\in(0,\frac{\pi}{2})$ if its region of absolute stability (RAS) contains the infinite wedge w_α, $w_\alpha=\{\lambda h:-\alpha\le|\pi-\arg(z)|\le\alpha\}$

Following the analysis as above, the order 10 of the constructed method is $A(\alpha)-Stable$ with $\alpha=79.75^o$ and therefore $L(\alpha)-Stable$

4 Numerical Experiments

In this section, we considered two problems to test the effectiveness of the method

Problem 1: Van der Pol problem (cf: [5])

$$y_1' = y_2$$
$$y_2' = -y_1 + \mu\, y_2(1 - y_1^2);\ y_1(0) = 2,\ y_2(0) = 0, \mu = 200$$

The phase diagram of the problem of the computed solution and that of ode15s are plotted in Fig. 1 and they produced the graph.

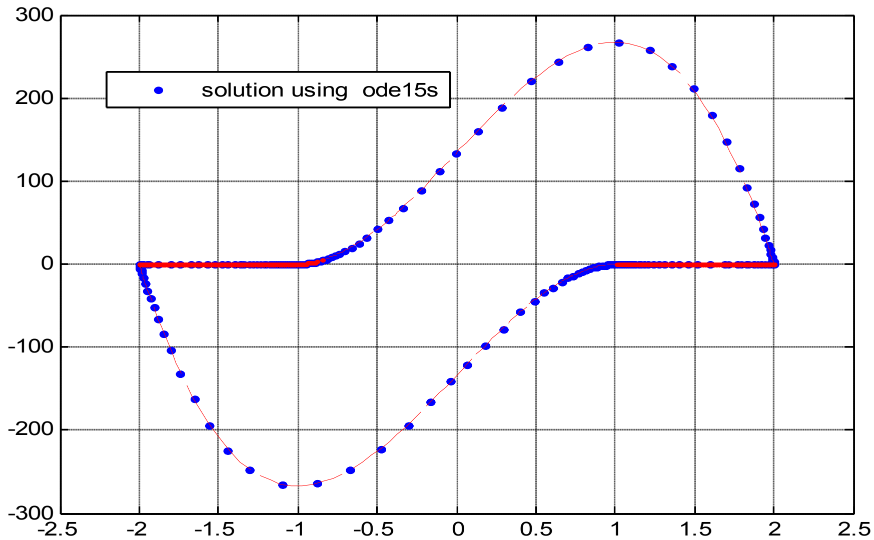

Fig. 1. The phase diagram of problem computed with order 6 of the method

Problem 2:

Consider the following linear constant coefficient initial value problem taken from [5],

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y; \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The theoretical solution is given by

$$y(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t)) \\ e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t)) \\ 2e^{-40t}(\sin(40t) - \cos(40t)) \end{pmatrix}$$

The exact and the computed solution using order 6 of our new method produced the same graph as can be seen in the Fig. 2.

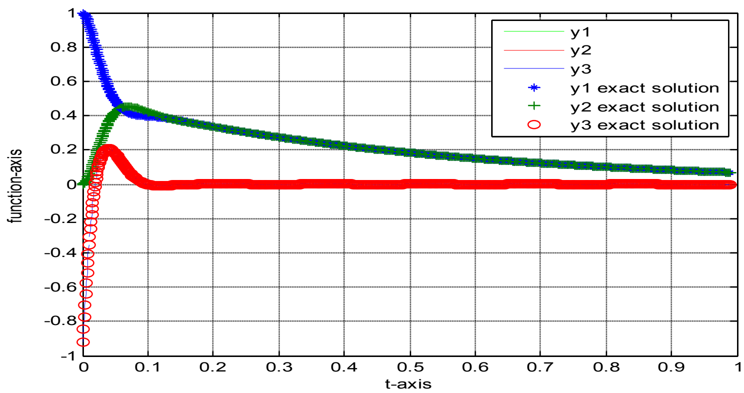

Fig. 2. Solution of problem 2 using order p=6

5 Conclusion

The work done in [4,5,7] using linear multistep methods has been extended to multi-block methods. The order 6 of the methods constructed is $L-Stable$, while the order 10 is $L(\alpha)-Stable$ with $\alpha = 79.75^o$. The result of the implementation of order 6 of the method on a stiff initial value problem shows that it is effective.

Competing Interests

Authors have declared that no competing interests exist.

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