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Tunneling-Induced Decay of Wave Packets Localized Beyond an Asymmetric Potential Barrier

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

The time behavior of a wave packet, transmitted through a potential barrier, is investigated for all times assuming it to be initially located outside the asymmetric barrier. The survival probability of the initial state, which characterizes the decay of the unstable quantum system in three time domains, is discussed. The presence of initial, exponential, and long-time regimes is a general feature of the decay process. We analytically found the three crucial points of time that separate these regimes. These characteristic times are set by simple formulae, including momentum distribution. It is then shown that the radius of convergence of the cumulant expansion for the survival probability is always small. A long-time behavior of survival probability is described by the asymptotic series whose coefficients decrease with increasing particle energy but at different rates. To be specific, we have examined the exactly solvable model of tunneling through a finite range triangular barrier.

Keywords: Time-dependent tunneling; survival probability; wave packet; cumulant expansion; triangular potential barrier.

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1. INTRODUCTION

The decay of unstable quantum systems has always attracted much attention of a physical community, starting from the early days of quantum mechanics. Quantum decays of isolated unstable systems appear in a wide variety of fields ranging from quantum physics [1-5], statistical mechanics [4,5], and chemistry [6] to cosmology [7]. While the exponential decay law occurs everywhere in nature, microscopic systems should be described by quantum mechanics and the exponential decay cannot be valid for all times. In particular, it undoubtedly fails when the time of evolution is either long [3,8-10] or short [9-12]. The exponential decay law is commonly associated with the survival probability that a system, initially prepared in an unstable state, will be still in the same state after some time [8-10]. The survival probability yields a quadratic behavior at short times, exponential law at intermediate times, and power law at longer times [1-12]. The above-mentioned features of time evolution are the consequences of rather general considerations assuring that the quantum decay law can be described by this three-step function.

Much attention has been lately paid to the asymptotic behavior of wave packets, moving in a free space [13-19], as a simple example of initial state decay. If the Gaussian wave packet is chosen as the initial state, the wave packet decreases asymptotically as $t^{-1/2}$. However, this is not necessarily valid for an arbitrary initial wave packet [13]. Actually, a decrease, being either slower [14,15] or faster than this law, can occur for the wave packet which vanishes at zero momentum [16]. In addition, a study was conducted on the transmission of the Gaussian wave packet through a potential barrier [17,18]. It has been shown that the solution behaves itself as $t^{-3/2}$ at a fixed distance and very long times. The next result [19] is that the wave packet, transmitted through the barrier, can asymptotically behave like $t^{-j/2}$ where $j =$ 1,2,...These facts pose us a simple question of how the characteristics of both the initial wave packet and the potential barrier affect the long time behavior of the decaying state. The present work gives satisfactory answers to this question.

The presence of initial, exponential, and long-time power law regimes appears to be a universal feature of the decay process [3,9]. The decay of unstable states is one of the most pervasive and studied phenomena in quantum physics; nevertheless, some elements of its theory remain obscure in many directions. Calculations are often performed with a definite aim to demonstrate the required behavior of the decaying state when everything needed to know for the calculation is the predetermined model. Unfortunately, unavailable often is the definition of either the time interval boundaries or the crucial points of time, when one of the periods ends and another starts.

In this work, we consider the behavior of the survival probability of the decaying state in every detail with a wave packet moving to the barrier from the region beyond it. This formulation of the problem differs fundamentally from the classical treatment of alpha decay and the time-dependent method of wave packets is the necessary tool to solve it. Although some elements of our study are well known, being widely spread in the literature, it is useful to consider the typical features of decaying states again. The nature of the exponential decay is the subject of wide speculation in the literature. With this in mind, we pay particular attention to the emergence of exponential decay; analyze approximations that justify it, and present criteria for its breakdown. We also perform a detailed investigation of non-exponential decay in the short- and long-time limits. The cumulant expansion is often used for this purpose. Its radius of convergence appears to be very small, a property which confines its domain of applicability. The asymptotic series of the survival probability works well at very long times and is valid for all sub-barrier energies of incident particles. The transition from one regime to another is usually accompanied by the cancelation of the corresponding survival probabilities, observed as oscillations in the decay curve. In this case, we give a qualitative and quantitative treatment of this phenomenon. Our conclusions are of general character and independent of the barrier form. To be specific, we have examined the exactly solvable model of tunneling through a finite-range triangular barrier. In the course of our discussion, we are going to use the computed solution to illustrate general conclusions by numerically exact calculations.

2. SURVIVAL PROBABILITY OF UNSTABLE STATES DUE TO TUNNELING

To formulate the problem of calculating the survival probability, consider first some relevant

results of the stationary and non-stationary descriptions of quantum tunneling. Let a particle be of mass m and move with momentum $\hbar k$ towards the potential barrier $V(x)$. We write the one-dimensional Schrödinger equation for scattering states

$$\frac{d^2\Phi}{dx^2}+\left[k^2-\frac{2m}{\hbar^2}V(x)\right]\Phi(x,k)=0 \qquad (1)$$

and assume that the real potential $V(x)$ vanishes sufficiently rapidly at $x\to\pm\infty$. The proper solutions to this equation are the two linearly independent wave functions $\Phi_L(x,k)$ and $\Phi_R(x,k)$, determined uniquely by boundary conditions [20,21]

$$\Phi_L(x,k)=\begin{cases} e^{ikx}+R_L(k)e^{-ikx}, & x\to-\infty \\ T(k)e^{ikx}, & x\to+\infty \end{cases}, \qquad (2)$$

$$\Phi_R(x,k)=\begin{cases} T(k)e^{-ikx}, & x\to-\infty \\ e^{-ikx}+R_R(k)e^{ikx}, & x\to+\infty, \end{cases} \qquad (3)$$

Where,

$T(k)$ is the transmission amplitude; $R_L(k)$ and $R_R(k)$ are the reflection amplitudes.

The wave functions Φ_L and Φ_R are further written down as

$$\Phi_0(x,k)=\frac{1}{\sqrt{4\pi}}\Phi_L(x,k), \quad \Phi_1(x,k)=\frac{1}{\sqrt{4\pi}}\Phi_R(x,k), \qquad (4)$$

The wave functions $\Phi_j(x,k)$, where $j=0,1$, are the fundamental system of solutions, i.e., the general solution can be given as a linear superposition of these functions.

It is assumed that a particle is described by a wave packet, i.e., the fully time-dependent method is applied to scattering on a stationary barrier. Consider now a large ensemble of the identically prepared single-particle scattering events, in which the particle with the same initial wave function, $\Psi(x,0)$, is incident on the barrier from the left

$$\Psi(x,0)=\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}}\exp\left[-\frac{(x-x_i)^2}{4\sigma^2}+ik_i x\right] \qquad (5)$$

Where.

σ^2 is the wave packet spatial variation. In other words, the wave packet, localized at $t=0$ near $x_i<0$ in front of the barrier, moves towards positive x -coordinates with the average momentum $\hbar k_i$ and the average energy $E=\left(\hbar^2/2m\right)\left(k_i^2+1/4\sigma^2\right)$. Since the time-dependent Schrödinger equation is linear in the derivative with respect to time, a specification of the wave function at the initial time is sufficient to determine this function at any future time. Thus, we expand the initial wave packet in terms of a complete set of scattering states

$$\Psi(x,0)=\int_{-\infty}^{\infty}\left[A_0(k)\Phi_0(x,k)+A_1(k)\Phi_1(x,k)\right]dk \qquad (6)$$

Where,

The weight amplitude $A_j(k)$ is of the form

$$A_j(k)=\int_{-\infty}^{\infty}\Psi(x,0)\Phi_j(x,k)^*dx, \quad j=0,1 \qquad (7)$$

The general solution is the superposition of partial solutions

$$\Psi(x,t)=\int_{-\infty}^{\infty}\left[A_0(k)\Phi_0(x,k)+A_1(k)\Phi_1(x,k)\right]\exp\left(-i\frac{\hbar k^2}{2m}t\right)dk \qquad (8)$$

The solution derived must satisfy the condition

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx=\int_{-\infty}^{\infty}\left(|A_0(k)|^2+|A_1(k)|^2\right)dk=1 \qquad (9)$$

Equations (6) and (9) can be used to verify the validity of all numerical calculations.

Let the system under consideration exist at $t=0$ in the $\Psi(x,0)$ state. The potential barrier

affects the motion of the system as perturbation and results in the $\Psi(x,t)$ state for the future time. Our prime interest here is with the decay of the initial state. This problem amounts precisely to the determination of the survival probability that the system at time t will be still in the unperturbed state. In the case of a closed system, the desired survival amplitude is the overlap integral of perturbed and unperturbed wave packets [8,9]

$$F(t) = \int_{-\infty}^{\infty} \Psi(x,0)^* \Psi(x,t) dx . \tag{10}$$

The square of its modulus is the survival probability $f(t)$. It is identical to the quantum fidelity between the initial state and the time-evolving state [22]. The $f(t)$ is also closely related to but different from the Loschmidt echo [4].

We now substitute Eq. (8) into (10) to obtain an explicit expression for the survival amplitude.

$$F(t) = \int_{-\infty}^{\infty} e^{-i\lambda(t)k^2} \varphi(k) dk , \tag{11}$$

where ,

$\lambda(t) = \hbar t / 2m$, and $\varphi(k)$ is the normalized probability distribution over wave numbers

$$\varphi(k) = \left|A_0(k)\right|^2 + \left|A_1(k)\right|^2 . \tag{12}$$

The first term describes the wave, running from left to right. It is this term that makes the main contribution for positive k -numbers. On the contrary, the second term corresponds to that, running from right to left, with the main contribution for negative k -numbers. In total, $\varphi(k)$ is the even function of k. Fig. 1 gives an idea of the typical behavior of $\varphi(k)$ as a function of the wave number k for the triangular barrier (see below Eq. (A.1)). If a particle starts from a position to the left of the barrier, then the distribution $|A_0(k)|^2$ is much larger than $|A_1(k)|^2$. This inequality is reversed when the particle is initially located to the right of the barrier. Detailed calculations are given in the Appendix.

Now consider the survival probability which contains three terms $f(t) = f_0(t) + f_1(t) + f_{\text{int}}(t)$ where the contribution of the momentum distribution $|A_j(k)|^2$ to the survival probability is

$$f_j(t) = \left| \int_{-\infty}^{\infty} e^{-i\lambda(t)k^2} \left|A_j(k)\right|^2 dk \right|^2 \tag{13}$$

and $f_{\text{int}}(t)$ gives the interference term

$$f_{\text{int}}(t) = 2\int_{-\infty}^{\infty} \left|A_0(k)\right|^2 dk \int_{-\infty}^{\infty} \left|A_1(k')\right|^2 \cos\left[\lambda(t)(k^2 - k'^2)\right] dk' \tag{14}$$

As follows, this is the alternating quantity due to the cosine in the integrand. Each of the momentum distributions $|A_j(k)|^2$ is not symmetric in the variable k, but the integrand is a symmetric function of the variables k and k'. Interference arises from the existence of two decay channels with the particles moving both towards and backwards the barrier. Figuratively speaking, the detector receives two waves from the different sources. In this case, the intensities interfere with each other rather than with the fields.

If the Hamiltonian has a purely continuous spectrum, then $f(\infty) = 0$. Mathematically, this conclusion follows from the Riemann-Lebesgue lemma. In this case, the system cannot be close to its initial state and actually, becomes orthogonal to it for a long time. Fig. 2 illustrates these considerations for the triangular barrier. Peaks on the curve are indicative of the process of partial regeneration of the initial state, because the system passes several times through the state characterized by the condition $f(t_m) = \varepsilon$ ($m = 1,2,...$), where $\varepsilon \neq 1$. As the wave packet tunnels through the barrier, a part of the incident packet interferes with a portion of the packet that has already been reflected. Therefore, the number of peaks depends on the x_i -distance.

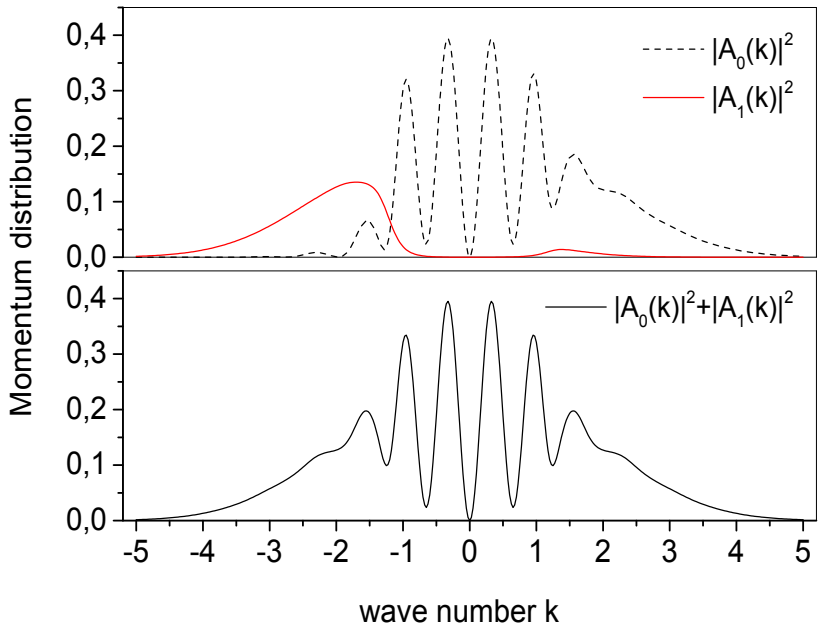

Fig. 1. The momentum distribution $\varphi(k)$ as a function of the wave number k. The triangular barrier parameters are V_0=1 and d=4. The Gaussian wave packet parameters are x_i=-4, σ=0.4, and k_i=1.2

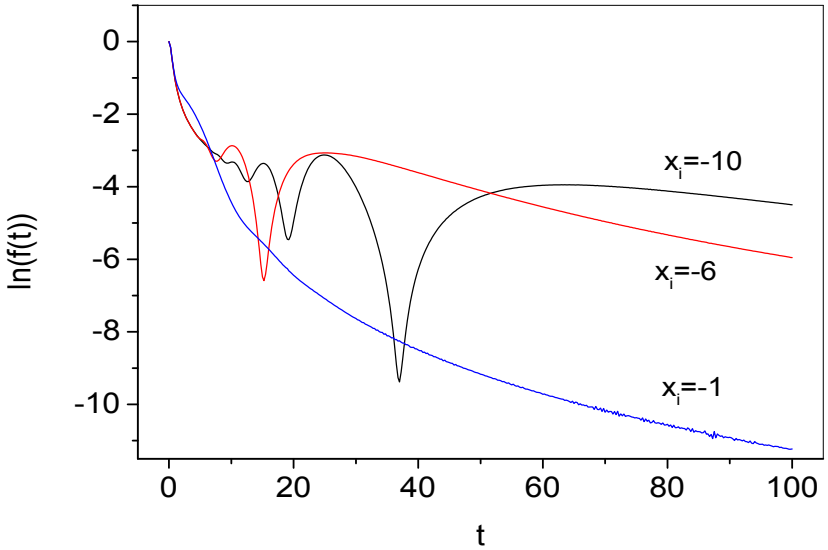

Fig. 2. Plot of $\ln(f(t))$ as a function of time t for the different values of the x_i-distance to the scattering center. The parameters are chosen as V_0=1, d=5, σ=0.4, and k_i=1

3. SHORT-TIME BEHAVIOR

Consider now the survival probability behavior in the short-time limit. To establish a relation between the survival probability and the moments of momentum distribution, it is sufficient to expand $\exp(-i\lambda(t)k^2)$ into a series of $\lambda(t)$, namely:

$$f(t) = \sum_{n,n'=0}^{\infty} (-i)^{n-n'} \frac{\lambda(t)^{n+n'}}{n!n'!} m_n m_{n'} \quad (15)$$

Where,

m_n are the moments of the $\varphi(k)$ momentum distribution

$$m_n = \int_{-\infty}^{\infty} k^{2n} \varphi(k) dk \,. \quad (16)$$

This expansion into the infinite double series converges absolutely for any $\lambda(t)$ value. We face the problem of convergence when accounting for the finite number of the terms of this expansion. In this case, the Taylor expansion of $\ln(f(t))$ works better than the Taylor expansion of $f(t)$ itself [23]. The cumulant expansion of $f(t)$ reads as

$$f(t) = \exp\left[2\sum_{n=1}^{\infty} (-1)^n \frac{\lambda(t)^{2n}}{(2n)!} \kappa_{2n} \right] \quad (17)$$

The cumulant coefficients κ_n are related to the moments m_n via the recursion formula [24]: $\kappa_1 = m_1$ at $n = 1$ and for $n \geq 2$, the coefficients are given by

$$\kappa_n = m_n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_j m_{n-j} \,. \quad (18)$$

The κ_n coefficients are determined analytically by this formula and can be calculated numerically up to very large n. Then, we rewrite Eq. (17) in the more convenient form

$$f(t) = \exp\left[\sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{\tau_{2n}} \right], \quad (19)$$

Where,

$$\tau_n = \frac{n!}{2(\hbar / 2m)^n \kappa_n} \,. \quad (20)$$

The values of variable t are observed to be less than the radius of convergence which equals $R = \tau_{2n}^{1/2n}$ at $n \to \infty$. Fig. 3 shows the dependence of $\tau_n^{1/n}$ on n. The value of τ_1 is equal to $m / \hbar m_1$ and $\sqrt{\tau_2}$ is given by $2\tau_1 m_1 / \sqrt{m_2 - m_1^2}$, and so on. According to numerical calculations, the radius of convergence is independent of the initial momentum $\hbar k_i$.

On the other hand, the radius of convergence grows with increasing localization length x_i of the wave packet and reaches its maximum at large x_i values (Fig. 4).

If we fix the parameters of both the potential and the wave packet but changing x_i, then we get the optimal x_i length at which R is maximum and the cumulant approximation holds well for $t < R$. The fact remains that the numerical calculations show that the radius of convergence is always small for any set of these parameters. In particular, the time $\sqrt{\tau_2}$ is generally very short compared to the lifetime, see e.g. [3,9]. As is clear from (19), the survival probability is a function of t^2, the property which excludes the possibility of exponential decay. Our analysis indicates that this statement is not the case. The cumulant expansion diverges at long times. Thus, a special study should be performed on the $f(t)$ time behavior beyond the validity range of the cumulant expansion.

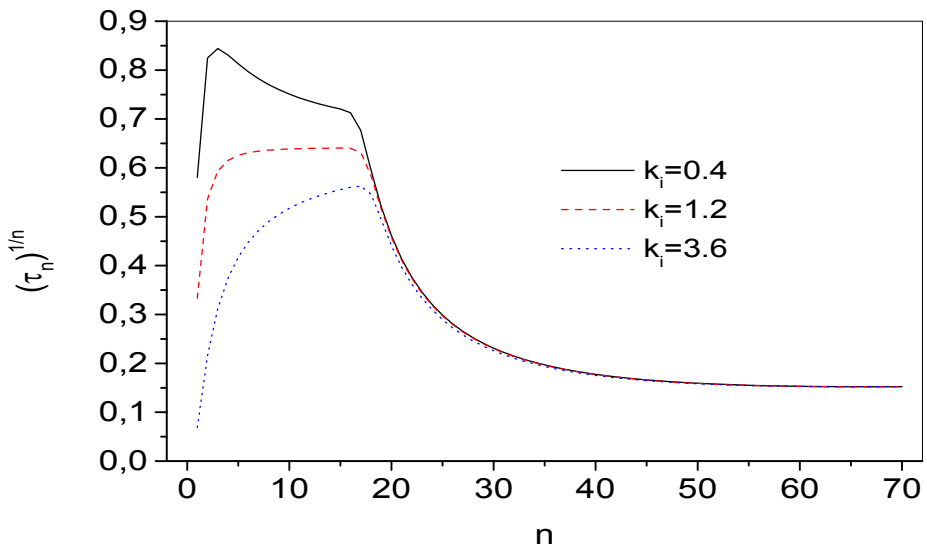

Fig. 3. Plot of $\tau_n^{1/n}$ versus n for the different values k_i of the initial momentum. The parameters are V_0=1, d=4, x_i=-4, and σ=0.4

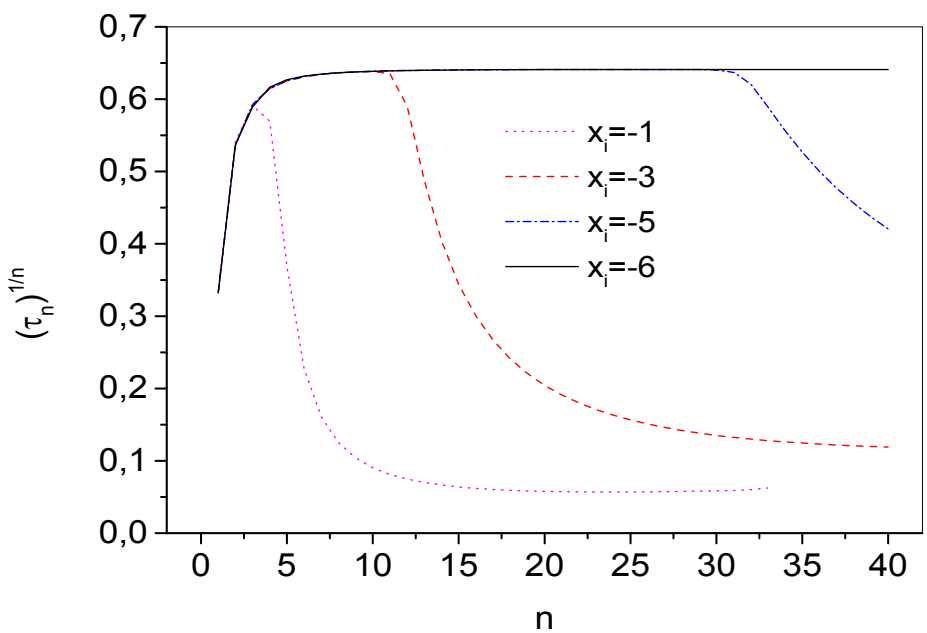

Fig. 4. Plot of $\tau_n^{1/n}$ versus n for the different values of the x_i -distance to the scattering center. The parameters are V_0=1, d=4, k_i=1.2, and σ=0.4

4. LONG-TIME BEHAVIOR

Now we consider the behavior of survival amplitude at long times. To this end, the function $F(t)$ is written as

$$F(t) = 2\int_0^\infty e^{X(k,t)}dk\,, \tag{21}$$

Where,

$X(k,t) = -i\lambda(t)k^2 + \ln(\varphi(k))$ is the even function k which depends on the large positive parameter $\lambda(t)$. The survival amplitude, in this case, is small due to the fast oscillations of the factor $\exp\left(-i\lambda(t)k^2\right)$. The most general result follows from the Riemann-Lebesgue lemma, which asserts that with $\lambda \to \infty$ the integral tends to zero. It is obvious that the main contribution to the integral is made by the stationary point at which the oscillations slow down. Here, we encounter a more complex case where the equations $X^{(1)}(k_0,t) = 0$ and $\mathrm{Re}\left(X^{(2)}(k_0,t)\right) < 0$ have no solution to all t.

Thus, the stationary point can be determined from the condition

$$X^{(3)}(k_0) = 0 \quad and \quad X^{(4)}(k_0) < 0\,, \tag{22}$$

Where,

k_0 is the real-valued positive root. In this case, the method of stationary phase holds because these derivatives are already time-independent. From now on, we use the dimensionless variable $(k - k_0)/k_0 = \xi$. The evaluation of (21) is reduced to the calculation of the approximate expression.

Further, we are going to carefully consider the dependence of the derivatives, included in Eq. (23), on the variables k_0 and t. To simplify notations, we take the following $a(t)$, $b(t)$, and c as dimensionless quantities:

$$F(t) = 2k_0 e^{X(k_0,t)} \times \int_{-1}^{\infty} \exp\left[X^{(1)}(k_0,t)k_0\xi + X^{(2)}(k_0,t)\frac{k_0^2\xi^2}{2} + X^{(4)}(k_0)\frac{k_0^4\xi^4}{24}\right]d\xi \tag{23}$$

$$a(t) = X^{(1)}(k_0,t)k_0, \quad b(t) = X^{(2)}(k_0,t)\frac{k_0^2}{2}, \quad c = -X^{(4)}(k_0)\frac{k_0^4}{24}\,. \tag{24}$$

Evaluating the derivatives in Eq. (24) yields:

$$a(t) = \frac{t_a + it}{t_b} \quad and \quad b(t) = \; = \frac{t_c + it}{2t_b}\,, \tag{25}$$

Where,

$$t_a = t_b\left[-\frac{\varphi^{(1)}(k_0)k_0}{\varphi(k_0)}\right], \tag{26}$$

$$t_b = \frac{m}{\hbar k_0^2}\,, \tag{27}$$

$$t_c = t_b\left[-\frac{\varphi^{(2)}(k_0)k_0^2}{\varphi(k_0)} + \left(\frac{\varphi^{(1)}(k_0)k_0}{\varphi(k_0)}\right)^2\right]. \quad (28)$$

Thus, in our theory, the three relevant time scales appear to which the system evolution time t is compared. It is worth noting that the function $a(t)$ is weakly dependent on time at $t << t_a$, while the $b(t)$ function is time-independent at $t << t_c$. The hierarchy of these time scales $t_a << t_b << t_c$ is to be discussed later.

Now we rewrite Eq. (23) in the new notations

$$F(t) = 2k_0 e^{X(k_0,t)} \int_{-1}^{\infty} \exp\left[Y(\xi,t)\right] d\xi, \quad (29)$$

Where,

$$Y(\xi,t) = -a(t)\xi - b(t)\xi^2 - c\xi^4. \quad (30)$$

A careful analysis indicates that the solutions of equations $Y^{(1)}(\xi_0,t) = 0$ and $\mathrm{Re}\left(Y^{(2)}(\xi_0,t)\right) < 0$ hold for all t. Therefore, we use the usual method of stationary phase to evaluate $F(t)$. The stationary point ξ_0 is found from the following cubic equation

$$Y^{(1)}(\xi_0,t) = -a(t) - 2b(t)\xi_0 - 4c\xi_0^3 = 0. \quad (31)$$

From the three roots of the equation we need to choose one whose imaginary part is negative. If there are two such roots, the one with the minimal imaginary part is taken. It should be verified that $\mathrm{Re}\left(Y^{(2)}(\xi_0,t)\right) < 0$ for any t. Let us denote this root as $\xi_0 = \xi_0' + i\xi_0''$. The $F(t)$ is obtained immediately as

$$F(t) = 2k_0 \exp\left[X(k_0,t) + Y(\xi_0,t)\right]\left[\frac{2\pi}{-Y^{(2)}(\xi_0,t)}\right]^{\frac{1}{2}} \quad (32)$$

With further algebraic manipulations, it is not difficult to show that Eq. (32) leads to the following expression for $f(t)$:

$$f(t) = 8\pi k_0^2 \varphi(k_0)^2 Z(t) \exp\left(-\frac{t}{t_d}\right), \quad (33)$$

Where,

$$t_d = \frac{t_b}{2\left(1+\xi_0'\right)\left(-\xi_0''\right)} \quad (34)$$

$$Z(t) = \frac{\exp\left\{-2\xi_0'\frac{t_a}{t_b} - \left(\xi_0'^2 - \xi_0''^2\right)\frac{t_c}{t_b} - 2c\,\mathrm{Re}\left(\xi_0^4\right)\right\}}{\left[\left(\frac{t_c}{t_b} + 12c\left(\xi_0'^2 - \xi_0''^2\right)\right)^2 + \left(\frac{t}{t_b} + 24c\xi_0'\xi_0''\right)^2\right]^{\frac{1}{2}}} \quad (35)$$

Equations (33)-(35) tell us that the survival probability is approximately exponential in form for not too small and not too large values of t. There is an enormous amount of literature on this delicate subject (see, for example, [1-6] and references therein). As is easy to see, the $f(t)$ probability behaves as $1-\lambda(t)^2\kappa_2$ for very small t. This implies that $\dot{f}(0) = 0$ and, therefore, the decay law cannot be exponential in the $t \to 0$ limit. On the other hand, an exact exponential decay is obtained in the quantum theory by using the Fourier transform of the Breit-Wigner energy density distribution [8,9]. This BW distribution has nothing to do with the present probability distribution (see Fig. 1). As a result, the tunneling dynamics provides the exponential decay law only in the approximation of large $\lambda(t)$. Now we need to reveal the causes that lead to its limitations at both long and short times.

First, the characteristic time t_d depends on the ξ_0 root of the cubic equation with time-dependent coefficients. The t_d time, as a function of t, is shown in Fig. 5.

As expected, the lengthy time interval exists in reality. During this interval, the t_d is practically time-independent. This behavior of t_d versus t is quite understandable if we study the dependence of t_d on the ξ_0 root. Numerical calculations indicate that at $t < t_b$ the imaginary

part of the ξ_0 root decreases rapidly and tends to zero with $t \to 0$. On the other hand, at $t > t_c$ the real part of ξ_0 tends to -1 at long times. It may be concluded then that the exponential decay begins with $t \approx t_b$ and ends later than $t \approx t_c$. Surely, the lower and the upper bounds of the exponential decay are determined approximately as the characteristic scale quantities.

Second, the survival probability involves another time-dependent factor $Z(t)$. This is a very slowly decreasing function of time, and, therefore, the effect on the $f(t)$ probability is trivial.

Let us turn again to Fig. 5. As follows, beyond the validity of the exponential approximation, the t_d time starts to increase sharply with $t > t_c$. The survival probability decreases exponentially to zero and the present approach, developed for its estimation, fails. Actually, at very long times, the decay law changes from the exponential one to the inverse power law.

5. POWER DECAY OF THE SURVIVAL PROBABILITY AT VERY LONG TIMES

We next focus our attention on the following integral expression for $\exp(-i\lambda k^2)$:

$$e^{-i\lambda k^2} = \frac{e^{-i\pi/4}}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} \exp\left(i\frac{x^2}{4\lambda} + ikx \right) dx \quad (36)$$

where,

λ and k are the real-valued parameters. Formally, the integral can be represented as the series whose terms are expressed in terms of the derivatives of the delta-function:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ikx} \sum_{n=0}^{\infty} \left(\frac{ix^2}{4\lambda} \right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \left(\frac{-i}{4\lambda} \right)^n \frac{\delta^{(2n)}(k)}{n!} \quad (37)$$

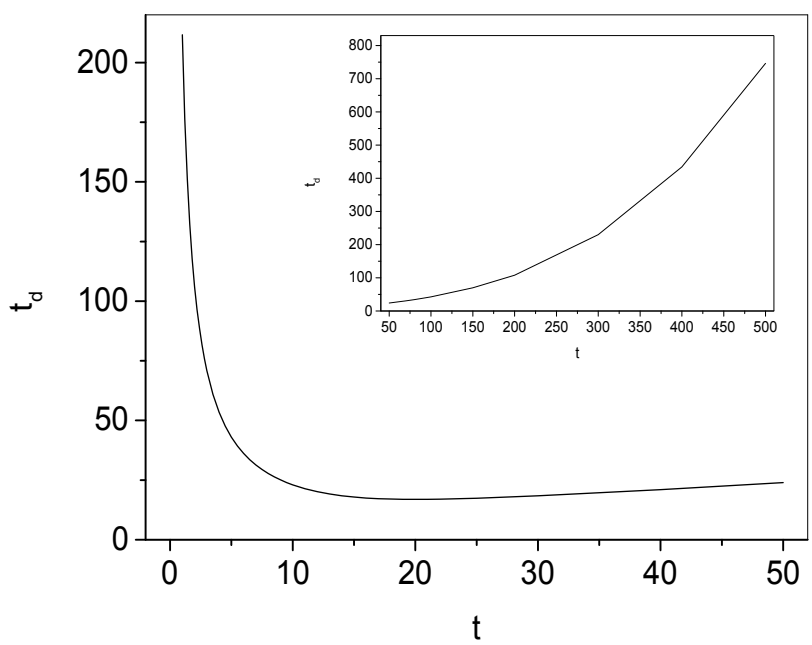

Fig. 5. A typical form of the time scale t_d versus t. The parameters are the same as in Fig. 1. The calculated characteristic times are t_a =0.916, t_b =8.94, t_c =45.2. The inset shows the breakdown of the exponential decay at very long times

The action of the generalized $\delta^{(2n)}(k)$ function on the $\varphi(k)$ one yields the solution to our problem. Equations (11) and (36), in conjunction with Eq. (37), are asymptotically equivalent to

$$F(t) = \sqrt{4\pi} e^{-i\pi/4} \sum_{n=1}^{\infty} \frac{(-i)^n \varphi^{(2n)}(0)}{n!(4\lambda(t))^{n+1/2}} \quad (38)$$

As one might expect, there is a strong correlation between the behavior of the momentum distribution in the vicinity of zero momentum and the asymptotic temporal behavior of the survival amplitude. It is physically clear that the slowest particles survive if the moment of observation is large. In this asymptotic series, the term with $n = 0$ is absent as $\varphi(0) = 0$. The presence of this correlation has been observed earlier [16]. Therefore, from Eq. (38) we obtain the asymptotic formula for the survival probability

We see that one of the remarkable properties of the survival probability is its power decay law for very long times. The words "very long times" mean that the power decay occurs only beyond the applicability limits of the exponential decay when the t_d quantity, as a function of time, increases sharply with t. The power decay was predicted for the systems with a continuous energy spectrum bounded from below [3,8]. As follows from Eq. (11), there are no limitations on the magnitude of the wave vector k, but the initial energy $E \geq E_{th} = \hbar^2 / 8m\sigma^2$ is bounded below. A particle can move both towards the barrier (with $k > 0$) and in the opposite direction (with $k < 0$). In addition, the two cases should be always considered for an asymmetric barrier where the particles move to the barrier both to the left and right which is taken into account by two terms in Eq. (12).

Let us now pay attention to a particular power law t^{-3} in Eq. (39). This asymptotic decay has been earlier determined for several models [9,14,17], and derived analytically [19]. We have performed numerical calculations for a number of potential barriers, including rectangular system (one- and double-barriers) [25], sech^{2}barrier [26], finite-range triangular barrier (see below), and generalized symmetric Woods-Saxon potential [27]. According to our calculations, the difference between the energy of the initially prepared wave packet and its minimum energy (denoted by $E_i = E - E_{th} = \hbar^2 k_i^2 / 2m$) is the controlling factor which determines the appearance of such a universal decay. The power decay law t^{-3} holds for all sub-barrier energies ($E_i < V_0$ where V_0 is the barrier height) independently of the barrier shape although the effect depends numerically on the value of the second derivative in (39). When the over-barrier energies are high ($E_i >> V_0$), the second derivative vanishes. With even higher energies, the fourth derivative tends to zero. The aforesaid means that the exponential decay regime is extended to longer times, and in expansion (39), the higher order terms in $\lambda(t)^{-2n-1}$ begin to play a significant role. If the post-exponential decay is not reached completely, then the first few terms contribute to the probability (39). Then the resulting curve can be approximated by the dependence $t^{-\mu}$, where μ is the non-integer number. This is the same dependence that was determined experimentally in [28], where the long-term decay of organic molecules luminescence upon pulsed laser excitation was measured. These results are assigned to the behavior of momentum distribution $\varphi(k)$, which consists of the inverted plateau near $k \approx 0$ at $E_i >> V_0$. Fig. 6 shows that the momentum distribution undergoes a quality change with increasing energy E_i, when the particle encounters the barrier as a vanishingly small perturbation.

$$f(t) = \frac{\pi}{16\lambda(t)^3}\left[\varphi^{(2)}(0)\right]^2 + \frac{\pi}{1024\lambda(t)^5}\left\{\left[\varphi^{(4)}(0)\right]^2 - \frac{4}{3}\varphi^{(2)}(0)\varphi^{(6)}(0)\right\} + \ldots \quad (39)$$

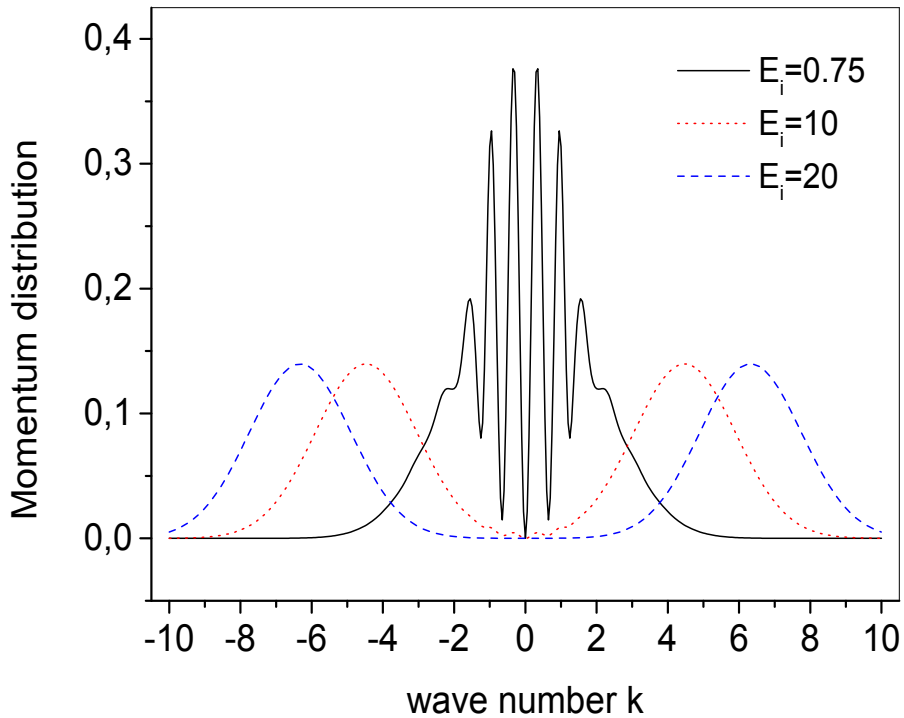

Fig. 6. The momentum distribution $\varphi(k)$ as a function of the wave number k for the different values of the E_i energy. The parameters are the same as in Fig. 1

6. CONCLUSIONS

In the present work, we have calculated the survival probability of unstable states, originating from the decays that can be understood in terms of tunneling. It is determined by momentum distribution and simple formulae. We have used a recursion formula to calculate cumulant coefficients and found the convergence radius of the cumulant expansion. The study of time evolution of the survival probability has been shown that there are three characteristic times t_a, t_b and t_c, on which the behavior of $f(t)$ is qualitativly different in four regions separated by these crucial points of time. At times shorter than t_a the decay regime can be described in terms of the cumulant expansion. The short-time limit can be sensitive to the details of the preparation of the initial wave packet, mainly, due to the particle momentum k_i and the distance x_i. The transient process, observed between t_a and t_b, is characterized by the essentially oscillatory decay curve. We have given a qualitative and quantitative explanation of oscillations in the decay curve. The demonstration of the exponential decay, occurring at a long time between t_b and t_c, was also shown. The length of the exponential part of the curve, $t_c - t_b$, increases with increasing distance x_i up to the scattering center. At a time longer than t_c, we have derived the power decay law. In this case, the survival probability is expressed in terms of an asymptotic series. The power decay holds for all sub-barrier energies independently of the barrier form. The problem of scattering on the finite-range triangular barrier was solved. Solutions were found for the particles, moving towards the barrier on the left and right. The transmission-reflection amplitudes were determined. The momentum distribution is sensitive to all essential details of tunneling and is calculated only in the presence of a complete

solution of this problem. Finally, our conclusions and results follow directly from “the first principles” ,i.e., from the Schrödinger equation.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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APPENDIX

TUNNELING THROUGHT THE TRIANGULAR POTENTIAL BARRIER

Here we are considering the exactly solvable model of a triangular potential barrier to exemplify a decay process in the tunneling phenomena realistic in atomic physics (electric-field induced tunneling).The analytic representation of potential barrier is of the form

$$V(x)=\begin{cases} 0,\ x<0, \\ V_0\left(1-\dfrac{x}{d}\right),\ 0\le x\le d, \\ 0,\ x>d, \end{cases} \tag{A.1}$$

Where,

V_0 is the barrier height, and d is the barrier length. The fundamental solutions to the Schrödinger equation with a linear potential can be expressed in terms of the Airy functions [29]. Taking into account Eqs. (2) and (3), we write down these solutions

$$\Phi_L(x,k)=\begin{cases} e^{ikx}+R_L(k)e^{-ikx},\ x<0, \\ CL_1(k)Ai\left(\alpha(k)-\beta x\right)+CL_2(k)Bi\left(\alpha(k)-\beta x\right),\ 0\le x\le d, \\ T(k)e^{ikx},\ x>d, \end{cases} \tag{A.2}$$

$$\Phi_R(x,k)=\begin{cases} T(k)e^{-ikx}, \ x<0, \\ CR_1(k)Ai\left(\alpha(k)-\beta x\right)+CR_2(k)Bi\left(\alpha(k)-\beta x\right), 0\le x\le d, \\ e^{-ikx}+R_R(k)e^{ikx}, \ x>d, \end{cases} \tag{A.3}$$

Where,

$$\alpha(k)=\left(\frac{2md^2V_0}{\hbar^2}\right)^{\frac{1}{3}}\left(1-\frac{\hbar^2k^2}{2mV_0}\right) \quad and \quad \beta=\frac{1}{d}\left(\frac{2md^2V_0}{\hbar^2}\right)^{\frac{1}{3}}. \tag{A.4}$$

The continuity condition for the wave function $\Phi_L(x,k)$ and its derivative at points $x=0,\ d$ provide four equations with four unknowns. Actually, we need to solve step-by-step the two systems of equations where each system consists of two linear equations. The same holds for the $\Phi_R(x,k)$ function. To avoid cumbersome formulas, it is convenient to define two auxiliary functions:

$$v_A(x,k)=ikAi\left(\alpha(k)-\beta x\right)+Ai'\left(\alpha(k)-\beta x\right), \tag{A.5}$$

$$v_B(x,k)=ikBi\left(\alpha(k)-\beta x\right)+Bi'\left(\alpha(k)-\beta x\right). \tag{A.6}$$

The other three auxiliary functions are presented as combinations of the above functions, taken at points 0 and d:

$$u_1(k)=v_A(0,k)v_B(d,k)-v_A(d,k)v_B(0,k), \tag{A.7}$$

$$u_2(k)=v_A(0,k)v_B(d,k)^*-v_A(d,k)^*v_B(0,k), \tag{A.8}$$

$$u_3(k)=\mathrm{Im}\left[v_A(0,k)v_B(0,k)^*\right]. \tag{A.9}$$

Simple calculations lead to the following expression for the transmission amplitude

$$T(k)=e^{-ikd}\frac{\left|u_1(k)\right|^2-\left|u_2(k)\right|^2}{2iu_2(k)u_3(k)}. \tag{A.10}$$

For the asymmetric barrier, the reflection amplitudes differ from each other:

$$R_L(k)=-\frac{u_1(k)^*}{u_2(k)} \quad and \quad R_R(k)=-e^{-2ikd}\frac{u_1(k)}{u_2(k)}. \tag{A.11}$$

It is clear that the reflection probabilities $\left|R_L(k)\right|^2$ and $\left|R_R(k)\right|^2$ are identical. Finally, the expansion coefficients obey the expressions:

(i) for the left-moving particles

$$CL_1(k) = \frac{k}{u_3(k)}\left[v_B(0,k)^* - \frac{u_1(k)^*}{u_2(k)} v_B(0,k) \right], \tag{A.12}$$

$$CL_2(k) = \frac{k}{u_3(k)}\left[-v_A(0,k)^* + \frac{u_1(k)^*}{u_2(k)} v_A(0,k) \right], \tag{A.13}$$

(ii) for the right-moving particles

$$CR_1(k) = \frac{ke^{-ikd}}{u_3(k)}\left[v_B(d,k) - \frac{u_1(k)}{u_2(k)} v_B(d,k)^* \right], \tag{A.14}$$

$$CR_2(k) = \frac{ke^{-ikd}}{u_3(k)}\left[-v_A(d,k) + \frac{u_1(k)}{u_2(k)} v_A(d,k)^* \right]. \tag{A.15}$$

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