Physical Science International Journal

24(4): 24-31, 2020; Article no.PSIJ.57293

ISSN: 2348-0130

Energy Spectrum for Gravitational Waves

W. F. Chagas-Filho[1*]

[1] *Department of Physics, Federal University of Sergipe, Brazil.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/PSIJ/2020/v24i430186

Editor(s):

(1) Prof. Shi-Hai Dong, National Polytechnic Institute, Mexico.

(2) Dr. Roberto Oscar Aquilano, National University of Rosario (UNR), Argentina.

Reviewers:

(1) Sie Long Kek, Universiti Tun Hussein Onn, Malaysia.

(2) Pipat Chooto, Prince of Songkla University, Thailand.

(3) Anton A. Lipovka, Universidad de Sonora, Mexico.

(4) Pasuspuleti Venkata Siva Kumar, India.

(5) Michael Ugwu Onuu, Alex Ekwueme Federal University, Nigeria.

Complete Peer review History: http://www.sdiarticle4.com/review-history/57293

Received 25 March 2020

Accepted 01 June 2020

Published 17 June 2020

Original Research Article

ABSTRACT

Loop Quantum Gravity (LQG) is a formalism for describing the quantum mechanics of the gravitational field based on the canonical quantization of General Relativity (GR). The most important result of LQG is that geometric quantities such as area and volume are not arbitrary but are quantized in terms of a minimum length. In this paper we investigate the possibility of combining the notion of a minimum length with the basic equations that describe wave propagation. We find that the minimum length, combined with the constancy of the speed of light, induces a natural spectrum for the energy of a gravitational wave.

Keywords: Constrained Hamiltonian systems; electrodynamics; general relativity; quantum gravity.

1 INTRODUCTION

With the detection of gravitational waves in 2015 no more room was left for doubts about the validity of General Relativity (GR) as the fundamental theory for the gravitational interaction. The detection of gravitational waves also confirms the point of view that has been advocated by some authors [1,2,3] that the theory describing the quantum mechanics of the gravitational interaction should be based on the quantization of GR.

**Corresponding author: E-mail: wfilho@ufs.br;*

GR is based on the invariance of the theory under space-time diffeomorphisms. As a consequence of this invariance, in GR the space and time coordinates have no intrinsic physical meaning. Physical predictions of GR are independent of these coordinates. Only predictions based on relations between observable quantities have a physical meaning in GR.

Today our mathematical description of gravitational waves propagating in space-time is only approximated and involves the assumption that the gravitational field is weak. Under this assumption GR can be turned into a linear theory by expanding the curved space-time metric $g_{\mu\nu}(x)$ around the flat Minkowski metric $\eta_{\mu\nu}$ according to

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad (1)$$

where $h_{\mu\nu}(x)$ is a small deviation from the flat metric. In the description of the propagation of gravitational waves in the linear theory a necessary step is to impose gauge conditions that eliminate the unphysical degrees of freedom of the theory. In ref. [4] the gravitational analogue of the Coulomb, or radiation gauge, is used to eliminate the unphysical degrees of freedom contained in the perturbation $h_{\mu\nu}(x)$. Some authors prefer to denote the gravitational analogue of the Coulomb gauge as the transverse-traceless gauge [5,6]. In Electrodynamics in flat space-time the Coulomb gauge is given by $\partial^a A_a = 0$, where $a = 1, 2, 3$, ∂^a is the ordinary partial derivative associated to the Minkowski metric $\eta_{\mu\nu}$ and A_a is the vector potential. In Electrodynamics in a curved space-time with metric $g_{\mu\nu}(x)$ the Coulomb gauge is given by $D^a A_a = 0$, where D^a is the covariant derivative associated to the metric $g_{\mu\nu}(x)$.

The most developed formalism for describing the quantum mechanics of the gravitational field based on GR is called Loop Quantum Gravity (LQG). LQG is based on the canonical quantization of GR using a connection formalism [7] that can be constructed after the Arnowitt, Deser and Misner (ADM) [8] splitting of space-time into space and time is performed. Although the original construction of LQG only works in space-time dimension $d = 4$, the formalism has now been extended to an arbitrary number of space-time dimensions [9,10,11,12]. The most important result of LQG is that geometric quantities, such as area and volume, cannot have arbitrary values in a gravitational field. Instead, areas and volumes are quantized in terms of a minimum length [1,3]

$$l_0 = \sqrt{8\pi\gamma} L_P \quad (2)$$

where γ is the Immirzi parameter [13], a small constant that fixes the precise scale of the quantum theory, and $L_P = 1,62 \times 10^{-35} m$ is the Planck length. The eigenvalues of the area operator, for instance, are given in terms of this minimum length by [1,3]

$$A_j = l_0^2 \Sigma_i \sqrt{j_i(j_i + 1)} \quad (3)$$

where $j_i = \frac{1}{2}, 1, ...$are the spins of the corresponding link. The eigenvalues of the volume operator are also given in terms of the minimum length l_0.

In ref. [14] it was shown that, by using a Hamiltonian duality transformation in the phase space of LQG, it is possible to construct a classical first-order formalism which yields a quantum theory that can be interpreted as LQG in the momentum representation. One interesting feature of this classical formalism is that it has an $SU(2)$ and diffeomorphism invariant generalization of the Coulomb gauge of Electrodynamics in flat space as one of its basic equations. $SU(2)$ is the special unitary group, consisting of the symmetry transformations generated by complex 2×2 matrices that have determinant equal to 1. In addition, in the transition to the quantum theory, the generalized Coulomb gauge becomes a restriction on the wave functionals of the theory in the momentum representation. These features can be interpreted as suggesting that gravitational waves can propagate in gravitational fields which are not weak. This suggestion has been confirmed in modern treatments of GR using numerical simulations. For a discussion of strong sources of gravitational waves see ref. [5].

Based on the features mentioned above, it becomes interesting to investigate if there is any possibility of combining the prediction of LQG about the existence of a minimum length with the propagation of gravitational waves in strong gravitational fields, despite the fact that presently we have only numerical simulations indicating the existence of such waves. In this paper we present an initial step in the direction of this investigation. For physical reasons we select a particular value of the Immirzi parameter γ. The minimum eigenvalue of the area operator occurs for $j = \frac{1}{2}$ and is given by

$$A_{\min} = 4\sqrt{3}\pi\gamma L_P^2 \tag{4}$$

We choose $\gamma = (4\sqrt{3}\pi)^{-1}$ which gives $A_{\min} = L_P^2$. Therefore in this paper the minimum length that defines all physically possible areas and volumes in a gravitational field is the Planck length L_P.

This paper is organized as follows. After this first section of introduction, in section two we review the Hamiltonian formalism for Electrodynamics as a constrained system. We point out in this section that there are only two physical degrees of freedom in the configuration variable for Electrodynamics. We also point out that imposing the Lorentz gauge condition on the configuration variable does not eliminate the unphysical degrees of freedom and that the correct gauge condition is the Coulomb gauge. In section three we consider the basic classical and quantum equations for two complementary descriptions of quantum gravity. In this section we point out that the basic equations of GR in the Hamiltonian formalism automatically reduce the number of physical degrees of freedom in the configuration variable to two, the same number as in Electrodynamics. We also show how a generalized quantum version of the Coulomb gauge appears in quantum gravity. In section four we review the Planck scale. We think this revision is necessary in order to make the calculation in section five clear. In section five, using the ideas of the previous sections, we show how we can obtain an energy spectrum for a gravitational wave which propagates in a gravitational field defined on a space with a minimum length. Concluding remarks appear in section six.

2 ELECTRODYNAMICS AS A CONSTRAINED SYSTEM

Before considering the Hamiltonian formalism for GR, it is instructive to consider the Hamiltonian formalism for Maxwell´s Electrodynamics. We reproduce here, with some notational modifications for clarity, the treatment of Electrodynamics as a constrained Hamiltonian system, first presented by Dirac [15]. The Lagrangian, in Heaviside units, is written as

$$L = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^3x \tag{5}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

is the anti-symmetric tensor that describes the electromagnetic field and $A_\mu(x) = A_\mu(t, \vec{x})$, $\mu = 0, 1, 2, 3$ is the four-vector potential, treated here as the dynamic configuration variable of the system.

In order to make the transition to the Hamiltonian formalism we must define the canonical momentum conjugate to the configuration variable $A_\mu(x)$. Varying the velocity $\frac{\partial A_\mu}{\partial x^0} = \dot{A}_\mu$ in the Lagrangian (5) we obtain

$$\begin{aligned}\delta L &= -\frac{1}{2}\int F^{\mu\nu}\delta F_{\mu\nu}d^3x \\ &= \int F^{\mu 0}\delta \dot{A}_\mu d^3x \\ &= \int E^\mu \delta \dot{A}_\mu d^3x\end{aligned} \tag{7}$$

The canonical momentum is then given by

$$E^\mu(x) = F^{\mu 0}(x) \tag{8}$$

Now $F^{\mu\nu}$ is anti-symmetric

$$F^{\mu\nu} = -F^{\nu\mu}$$

and so if we set $\mu = 0$ in Equation (8) we obtain

$$E^0(x) = 0 \tag{9}$$

This is a primary constraint [15]. The other three components $E^a(x)$, $a = 1, 2, 3$ are the components of the electric field vector $\vec{E}(x)$.

Now we can compute the Hamiltonian H

$$\begin{aligned}H &= \int E^\mu \dot{A}_\mu d^3x - L \\ &= \int (F^{a0}\dot{A}_a + \frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}F^{a0}F_{a0})d^3x\end{aligned}$$

$$= \int (\frac{1}{4}F^{ab}F_{ab} - \frac{1}{2}F^{a0}F_{a0} + F^{a0}\partial_a A_0)d^3x$$
$$= \int (\frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}E^a E_a - A_0\partial_a E^a)d^3x \quad (10)$$

where an integration by parts was performed on the last term in the third line [15]. Now we must require the dynamic stability of the primary constraint (9). That is, we must require that

$$\{E^0(x), H\} = 0 \quad (11)$$

where $\{\ ,\ \}$ denotes the Poisson bracket on the phase space (A_μ, E^μ). Condition (11) can only be satisfied if

$$\partial_a E^a = 0 \quad (12)$$

Equation (12) is a secondary constraint [15]. It is the Gauss law for the electric field in the absence of electric charges.

Constraints (9) and (12) have vanishing Poisson brackets with themselves and with each other and are therefore first-class constraints [15]. The total Hamiltonian for Maxwell´s Electrodynamics can be written as [15]

$$H_T = \int (\frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}E^a E_a)d^3x$$
$$- \int A_0\partial_a E^a d^3x + \int \lambda E^0 d^3x \quad (13)$$

where $\lambda(t,x)$ is a Lagrange multiplier for the constraint $E^0 = 0$.

At this point we have four components for the vector potential and two first-class constraints. This means that we have only two physical degrees of freedom in the configuration variable. As we will see in the next section, this is the same number of physical degrees of freedom in the configuration variable for GR. This similarity in the number of physical degrees of freedom between the Hamiltonian formalisms for Electrodynamics and GR will be used later to support the ideas contained in section five.

The Hamilton equation of motion for A_0 gives

$$\dot{A}_0 = \{A_0, H_T\} = \lambda(t,x)$$

and we see that the time evolution of A_0 is completely arbitrary and therefore it is an unphysical variable. Furthermore, if we try to impose the Lorentz gauge condition

$$\partial_\mu A^\mu = 0$$

on the vector potential we get

$$\dot{A}_0 - \partial_a A^a = 0 \quad \rightarrow \quad \partial_a A^a = \lambda(t,x)$$

and we must conclude that the Lorentz gauge condition is not able to eliminate the remaining unphysical degree of freedom in the configuration variable. The correct gauge condition is therefore the Coulomb, or radiation gauge,

$$\partial_a A^a = 0 \quad (14)$$

In the next section we will see that a generalized version of the Coulomb gauge appears in quantum gravity as an operator equation on wave functionals. This will be interpreted as an evidence that gravitational waves also propagate in strong gravitational fields.

3 QUANTUM GRAVITY

After the ADM [8] splitting is performed, and the Ashtekar´s [7] connection variables $A_a^i(x)$ are introduced, GR can be cast as a constrained Hamiltonian system with first-class constraints. In this framework GR is described by the first-order action [1,3]

$$S = \frac{1}{8\pi i G}\int d^4x(E_i^a \dot{A}_a^i - \lambda^i D_a E_i^a - \lambda^a F_{ab}^i E_i^b - \lambda F_{ab}^{ij} E_i^a E_j^b) \quad (15)$$

Here $E_i^a(x)$ is the canonical momentum conjugated to the connection variable $A_a^i(x)$, $i,j = 1,2,3$ are internal $SU(2)$ indices and $a,b = 1,2,3$ are space indices.

$$D_a V^i = \partial_a V^i + \epsilon_{jk}^i A_a^j V^k \quad (16)$$

Equation (16) defines the covariant derivative on the tangent space of a three dimensional manifold Σ without boundaries,

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon_{jk}^i A_a^j A_b^k \quad (17)$$

and $F_{ab}^{ij} = \epsilon_k^{ij} F_{ab}^k$. The equations of motion for the Lagrangre multipliers λ^i, λ^a and λ give the first-class constraints

$$D_a E_i^a = 0 \quad (18)$$

$$F_{ab}^i E_i^b = 0 \quad (19)$$

$$F_{ab}^{ij} E_i^a E_j^b = 0 \quad (20)$$

Constraint (18) is an $SU(2)$ and diffeomorphism invariant generalization of the Gauss law constraint (12) we found in the Hamiltonian formalism for Electrodynamics. It generates internal $SU(2)$ gauge transformations in the curved space. Constraint (19) generates space diffeomorphisms and constraint (20) is the Hamiltonian constraint. There are nine degrees of freedom in the configuration variable A^i_a but there are seven first-class constraints. Therefore there are only two physical degrees of freedom in the configuration variable. As we saw in the previous section, this is the same number of physical degrees of freedom in the configuration variable for Electrodynamics. As we mentioned in the previous section, this similarity between Electrodynamics and GR in the Hamiltonian formalism will be used later to support the ideas in section five.

The transition to LQG is performed by promoting the canonical momenta E^a_i to quantum operators $\hat{E}^a_i = -i\hbar\frac{\delta}{\delta A^i_a}$ which act on configuration space wave functionals $\Psi(A)$. The constraints (18), (19) and (20) then become the quantum equations

$$D_a\frac{\delta}{\delta A^i_a}\Psi(A) = 0 \tag{21}$$

$$F^i_{ab}\frac{\delta}{\delta A^i_b}\Psi(A) = 0 \tag{22}$$

$$F^{ij}_{ab}\frac{\delta}{\delta A^i_a}\frac{\delta}{\delta A^j_b}\Psi(A) = 0 \tag{23}$$

Equations (21), (22) and (23) are the basic equations of LQG [1,3]. When these equations are valid the space has a minimum length L_P (for our choice of the Immirzi parameter γ).

Action (15) has a formal dual action that can be interpreted as describing the classical limit of a momentum space formulation of quantum gravity. This formal dual action is written as [14]

$$S = \frac{1}{8\pi iG}\int d^4x(-A^i_a\dot{E}^a_i - \lambda_i\bar{D}^aA^i_a - \lambda_a\bar{F}^{ab}_iA^i_b - \lambda\bar{F}^{ab}_{ij}A^i_aA^j_b) \tag{24}$$

where now

$$\bar{D}^aV^i = \partial^aV^i + \epsilon^{ijk}E^a_jV_k \tag{25}$$

defines the covariant derivative on the cotangent space of Σ,

$$\bar{F}^{ab}_i = \partial^aE^b_i - \partial^bE^a_i + \epsilon^{jk}_iE^a_jE^b_k \tag{26}$$

and $\bar{F}^{ab}_{ij} = \epsilon_{ijk}\bar{F}^{ab}_k$. The equations of motion for the Lagrange multipliers λ_i, λ_a and λ now give the constraints

$$\bar{D}^aA^i_a = 0 \tag{27}$$

$$\bar{F}^{ab}_iA^i_b = 0 \tag{28}$$

$$\bar{F}^{ab}_{ij}A^i_aA^j_b = 0 \tag{29}$$

Constraint (27) is a generalization of the Coulomb gauge (14) for Electrodynamics. While in the weak gravitational field limit of GR a considerable amount of work is required to reach the analogous of the Coulomb gauge (a necessary step to describe physical gravitational waves [4]) here we have a formal dual version of GR in which a generalized Coulomb gauge is one of its basic equations. Since the weak gravitational field limit was not required to arrive at constraint (27) we interpret this as an evidence that gravitational waves also propagate in gravitational fields which are not weak. As we mentioned in the introduction, this interpretation has already been confirmed in modern treatments of GR [5].

Now the transition to the quantum theory is performed by promoting the connection variables A^i_a to quantum operators $\hat{A}^i_a = i\hbar\frac{\delta}{\delta E^a_i}$ which act on momentum space wave functionals $\Psi(E)$. The constraints (27), (28) and (29) then become the quantum equations

$$\bar{D}^a\frac{\delta}{\delta E^a_i}\Psi(E) = 0 \tag{30}$$

$$\bar{F}^{ab}_i\frac{\delta}{\delta E^b_i}\Psi(E) = 0 \tag{31}$$

$$\bar{F}^{ab}_{ij}\frac{\delta}{\delta E^a_i}\frac{\delta}{\delta E^b_j}\Psi(E) = 0 \tag{32}$$

The quantum Equations (30), (31) and (32) can be interpreted as defining LQG in the momentum representation [14].

The quantum Equations (21), (22) and (23) in the configuration space and the quantum Equations (30), (31) and (32) in the momentum space are complementary and can be assumed to be simultaneously valid. When this is done, they open the possibility that gravitational waves can propagate in a gravitational field which is not weak and which is defined on a space with a minimum length.

4 THE PLANCK SCALE

In this section we review the Planck scale [16]. This is a necessary step in order to clarify the calculations to be presented in the next section. We start by considering the three fundamental constants of physics:

1) the speed of light in empty space

$$c = 2,998 \times 10^8 m/s$$

2) Newton´s gravitational constant

$$G = 6,67 \times 10^{-11} N.m^2.Kg^{-2}$$

3) Planck´s constant

$$\hbar = \frac{h}{2\pi} = 1,055 \times 10^{-34} J.s$$

In 1899 Planck [16] noticed that by combining these fundamental constants in a unique way he could define a fundamental scale of time, length and mass. Today this fundamental scale is called the Planck scale [17]. It is given by

1) the Planck time

$$T_P = \sqrt{\frac{\hbar G}{c^5}} = 5,40 \times 10^{-44} s \tag{33}$$

2) the Planck length

$$L_P = \sqrt{\frac{\hbar G}{c^3}} = 1,62 \times 10^{-35} m \tag{34}$$

3) the Planck mass

$$M_P = \sqrt{\frac{\hbar c}{G}} = 2,17 \times 10^{-5} g \tag{35}$$

Notice that the Planck length is the distance that light travels during the Planck time. Therefore the existence of a minimum length L_P, together with the constancy of the speed of light c, automatically leads to the existence of the Planck time T_P.

5 GRAVITATIONAL WAVES AND QUANTUM GRAVITY

As we saw in section three, combining the basic equations of quantum gravity in the configuration and in the momentum representation opens the possibility that gravitational waves can propagate in gravitational fields which are not weak and which are defined on spaces with a minimum length equal to the Planck length. Since at present time we have only an approximated description of gravitational waves, which is valid when the gravitational field is weak and the theory can be linearized, it is impossible to investigate here the detailed properties of a gravitational wave in the nonlinear theory. In the strong-gravity domain, reliable calculations are possible only through complex numerical simulations [5]. Nevertheless, those simulations show that many of the features that have been inferred about gravitational waves in the linear regime survive in some form in the nonlinear strong-gravity regime [5]. For this reason we think it may be possible to guess some simple physical assumptions about strong-gravity gravitational waves based on the behavior of weak-gravity gravitational waves, while at the same time incorporating the minimum length predicted by LQG. A plane wave, for instance, must have a well defined wavelength λ. We may then expect that the wavelength of a gravitational wave propagating in a space with a minimum length L_P should be an integer multiple of L_P, that is

$$\lambda_n = nL_P \qquad n = 1, 2, ... \tag{36}$$

A plane wave must also have a well defined period P. As we saw in section four, the existence of a minimum length L_P combined with the constancy of the speed of light leads to the existence of the Planck time T_P, which is the time necessary for light to travel the distance L_P. We may then expect that the period of a gravitational wave propagating in a space with a minimum length L_P should be an integer multiple of T_P, that is

$$P_n = nT_P \qquad n = 1, 2, ... \tag{37}$$

Equations (36) and (37) should be looked at with some care. This is because, as we mentioned in the introduction, as a consequence of diffeomorphism invariance, space and time coordinates have no intrinsic physical meaning in GR. However, Equations (36) and (37) do not involve any space or time coordinates. They are physical restrictions on the gravitational wave imposed by the existence of the minimum length and by the constancy of the speed of light. In other words, they are physical restrictions on the gravitational wave imposed by a quantum space. Therefore the restrictions contained in Equations (36) and (37) should lead, in principle, to observable effects. To confirm this idea we need to find an observable physical quantity associated with the gravitational wave. One such physical quantity is the energy of the gravitational wave. We know that electromagnetic waves and gravitational waves propagated with the same speed of light c in empty space. Further, as we saw in sections two and three, in the Hamiltonian formalism Electrodynamics and GR describe the same number of physical degrees of freedom. We may then use these two similarities between Electrodynamics and GR to support the idea that Planck´s equation for the energy of an electromagnetic wave, $E = h\nu$, should be valid also for a gravitational wave. Here E is the energy of the wave, $h = 2\pi\hbar$ with ħ the Planck constant given in section four and ν is the frequency of the wave.

Since the frequency ν of a wave is related to its period P as $\nu = P^{-1}$ we can combine Planck´s equation

$$E = h\nu \qquad (38)$$

with Equation (37) and write for a gravitational wave

$$E_n = h\nu_n = h\frac{1}{P_n} = h\frac{1}{nT_P} \qquad n = 1, 2, ...$$

Therefore we obtain

$$E_n = \frac{1}{n}\sqrt{\frac{2\pi h^2 c^5}{hG}}$$

which can be written as

$$E_n = \frac{2\pi c^2}{n}\sqrt{\frac{\hbar c}{G}} \qquad n = 1, 2, ... \qquad (39)$$

Equation (39) can be rewritten in a simple and elegant way as

$$E_n = \frac{2\pi}{n}E_{PE} \qquad n = 1, 2, ... \qquad (40)$$

where E_{PE} is a Planck-Einstein energy $E_{PE} = M_P c^2$ with M_P the Planck mass given in equation (35). We have therefore obtained a natural energy spectrum for gravitational waves. This spectrum is defined in terms of the speed of light c, Newton´s gravitational constant G, and Planck´s constant ħ

6 CONCLUDING REMARKS

In this paper, based on what can be considered as two complementary formulations of quantum gravity, we first pointed out an evidence for the fact that gravitational waves can propagate in gravitational fields which are not weak. This evidence is a gravitational generalization of the Coulomb gauge of electrodynamics. Using the notion of a minimum length predicted by LQG and assuming the validity of Planck´s equation $E = h\nu$ in the case of gravitational waves, we obtained a spectrum for the energy of a gravitational wave in terms of a Planck-Einstein energy $E_{PE} = M_P c^2$, where M_P is the Planck mass.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

[1] Rovelli C. Quantum gravity. Cambridge University Press; 2004.

[2] Rovelli C, Vidotto F. Covariant loop quantum gravity. Cambridge University Press; 2015.

[3] Thiemann T. Modern canonical quantum general relativity. Cambridge University Press; 2007.

[4] Wald RM. General relativity. The University of Chicago Press; 1984.

[5] Guidry M. Modern general relativity. Cambridge University Press; 2019.

[6] Carroll SM. Spacetime and geometry, an introduction to general relativity. Cambridge University Press; 2019

[7] Ashtekar A. New variables for classical and quantum Gravity. Phys. Rev. Lett. 1986;57:2244.

[8] Deser S, Arnowitt R, Misner CW. Consistency of canonical reduction of general relativity. New York: Belfer Graduate School of Science, Yeshiva University; 1964.

[9] Bodendorfer N, Thiemann T, Thurn A. New variables for classical and quantum gravity in all dimensions I. Hamiltonian Analysis, Class. Quant. Grav. 2013;30:045001. [arXiv:1105.3703 [gr-qc]]

[10] Bodendorfer N, Thiemann T, Thurn A. New variables for classical and quantum gravity in all dimensions II. Lagrangian Analysis, Class. Quant. Grav. 2013;30:045002. [arXiv:1105.3704 [gr-qc]]

[11] Bodendorfer N, Thiemann T, Thurn A. New variables for classical and quantum gravity in all dimensions III. Quantum Theory Class. Quant. Grav. 2013;30:045003. [arXiv:1105.3705 [gr-qc]]

[12] Bodendorfer N, Thiemann T, Thurn A. New variables for classical and quantum gravity in all dimensions IV. Matter Couplings, Class. Quant. Grav. 2013;30:045004. [arXiv:1105.3706 [gr-qc]]

[13] Immirzi G. Real and complex connections for canonical Gravity. Class. Quant. Grav. 1997;14:L177. [arXiv:gr-qc/9612030]

[14] Chagas-Filho WF. A First approach to loop quantum gravity in the momentum representation. Physical Science Inter. Jour. 2019;21(4):1-8.
DOI: 10.9734/PSIJ/2019/v21i430115

[15] Dirac PAM. Lectures on quantum mechanics. J. Math. Phys. 1960;1:434.

[16] Planck M. Uber irreversible Strahlungsvorgange, Sitzber kgl-preuβ, Akad. Wiss. Berlin, Sitzungen der phys-math, Klasse. 1899;440.

[17] Kiefer C, Quantum gravity. Oxford University Press; 2007.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle4.com/review-history/57293