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General Solutions and Generalized Hyer-Ulam Stability in Banach Spaces: A Direct Method Approach for a System of Functional Equations

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Authors' contributions

This work was carried out in collaboration among all authors. Author YG designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author LS and Author RP managed the analyses of the study. Author RP managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, we have obtained general solutions and demonstrated the generalized Hyer-Ulam stability for the following system of functional equations in Banach spaces using the direct method:

(i) $h(u_1 + u_2 + u_3) + h(u_1 + u_2 - u_3) + h(u_1 - u_2 + u_3) + h(u_1 - u_2 - u_3) = 4h(u_1),$

(ii) $h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) + h(3u_1 - 2u_2 - u_3) = 12h(u_1),$

(iii) $h(u_1 + 2u_2 + 3u_3) + h(u_1 + 2u_2 - 3u_3) + h(u_1 - 2u_2 + 3u_3) + h(u_1 - 2u_2 - 3u_3) = 4h(u_1).$

Keywords: Generalized hyers-ulam stability; additive functional equation; ulam stability; banach space.

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1 Introduction

The exploration of functional equation stability traces its origins back to S.M. Ulam [1] in 1940. Ulam posed a fundamental question: "Under what conditions can we confidently assert that a solution to a slightly modified equation remains in close proximity to the solution of the original equation?" This inquiry ignited a surge of research in the field. In 1941, D. H. Hyers [2] provided an affirmative response concerning the stability of Banach spaces, building upon Ulam's pioneering work. In 1950, T. Aoki [3] made substantial contributions to our understanding of additive mappings.

Th.M. Rassias [4] significantly reduced the prerequisites for the Cauchy difference through an extension of Hyers' Theorem. The contributions of Ulam, Hyers, and Rassias to the realm of stability concepts for functional equations have been thoroughly chronicled in the literature [5]-[10]. Their work played a pivotal role in advancing this area and eventually led to the establishment of Hyers-Ulam-Rassias stability.

In 1994, P. Gavruta [11] made advancements in the field by replacing the unbounded Cauchy difference with a control function of broader applicability, drawing inspiration from Rassias' approach. The aforementioned references provide a historical framework for examining the stability of functional equations and offer a comprehensive overview of existing research. Throughout the history of functional equations, mathematicians have explored solutions and stability across various categories of functional equations, as documented in diverse references [12]-[21].

In this section, we turn our attention to the examination of the stability of equations related to functional additivity within Banach spaces. Our primary focus is directed towards the scrutiny of the Cauchy functional equation:

$$K(g_1 + g_2) = K(g_1) + K(g_2). \tag{1.1}$$

It's noteworthy that the equation $h(s) = cs$ serves as a solution to the previously mentioned problem, representing what can be characterized as an additive functional equation. Several researchers have conducted analyses and published results related to Ulam-type stability for various additive functional equations using this framework.

Below, we present a selection of additive functional equations that have been explored in [22, 23, 24], along with their solutions and stability properties:

$$h(2g_1 - g_2) + h(g_1 - 2g_2) = 3h(g_1) - 3h(g_2), \tag{1.2}$$
$$h(2g_1 \pm g_2 \pm g_3) = h(g_1 \pm g_2) + h(g_1 \pm g_3), \tag{1.3}$$
$$h(g_1 + g_2 - 2g_3) + h(2g_1 + 2g_2 - g_3) = 3h(g_1) + 3h(g_2) - 3h(g_3), \tag{1.4}$$
$$\begin{aligned} &dh(g_1 + dg_2) - h(dg_1 + g_2) \\ &= \frac{d(d^2-1)}{2}[h(g_1 + g_2) + h(g_1 - g_2)] + (d - d^3)h(g_1) + (d^2 - 1)h(g_2). \end{aligned} \tag{1.5}$$

Recently, in their work, V. Govindan et al. [15] introduced a novel symmetric additive functional equation derived from a characteristic polynomial of degree three. This equation takes the following form:

$$\begin{aligned} &h[(a^3 + 11a)g_1 - 6(a^2 + 1)g_2] + h[(11a - 6a^2)g_2 + (a^3 - 6)g_3] \\ &- h[(a^3 - 6a^2)g_3 + (11a - 6)g_1] \\ &= (a^3 + 6)h(g_1) + (11a - 12a^2 - 6)h(g_2) + (6a^2 - 6)h(g_3), \end{aligned} \tag{1.6}$$

where $a \neq 0$. Their research encompassed a comprehensive examination of the entire solution space. Furthermore, they extended their study to assess the stability of this equation using Hyers-Ulam stability concepts, employing both fixed-point and direct methods in Banach spaces.

Numerous researchers have conducted thorough investigations into the stability of various functional equations, resulting in a wealth of fascinating discoveries. These findings have been extensively documented in recent references, including works such as Agilan and Rassias [25], Agilan et al. [26], Park and Rassias [27], and Pasupathi et al. [28], along with other significant contributions in this field.

The primary goal of this research is to propose an innovative form of a functional equation, which is presented as follows:

(i) $$\begin{aligned} &h(u_1 + u_2 + u_3) + h(u_1 + u_2 - u_3) + h(u_1 - u_2 + u_3) + h(u_1 - u_2 - u_3) \\ &= 4h(u_1), \end{aligned} \tag{1.7}$$

(ii) $$\begin{aligned} &h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) + h(3u_1 - 2u_2 - u_3) \\ &= 12h(u_1), \end{aligned} \tag{1.8}$$

(iii) $$\begin{aligned} &h(u_1 + 2u_2 + 3u_3) + h(u_1 + 2u_2 - 3u_3) + h(u_1 - 2u_2 + 3u_3) + h(u_1 - 2u_2 - 3u_3) \\ &= 4h(u_1) \end{aligned} \tag{1.9}$$

In this research paper, we employ a direct methodology to ascertain the solution of the given equation and explore its generalized Hyers-Ulam stability within the context of Banach spaces.

Within Section 2, we present an exposition that reveals the existence of comprehensive solutions for Equations (1.7), (1.8), and (1.9).

Moving forward to Section 3, we apply the Hyers' direct approach to assess the stability of the functional equation (1.8) within the generalized Hyers-Ulam criteria.

2 General Solutions

Equations (1.7), (1.8), and (1.9) will have their general solutions given in this section. In the following, we will refer to J and L as real vector spaces.

Theorem 2.1 ([13]). *If a mapping $h : J \to L$ satisfies the functional equation (1.1), $\forall u_1, u_2 \in J$, then the following properties hold:*

(i) $h(0) = 0$;

(ii) h *is an odd function;*

(iii) $h(\lambda u_1) = \lambda h(u_1)$ *for every number* $\lambda \in \mathbb{Q}$ *and for every* $u_1 \in J$.

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1,\, u_2 \in J$.
Setting $(u_1,\, u_2)$ to $(0,\, 0)$ in (1.1), we get $h(0) = 0$.
Furthermore, setting $(u_1,\, u_2)$ to $(0,\, u_1)$ in (1.1), we arrive at
$h(-u_1) = -h(u_1), \quad \forall\; u_1 \in J.$
Therefore, h is an odd function.
Replacing u_2 with u_1 and u_2 by $2u_1$ in (1.1), we respectively deduce
$h(2u_1) = 2h(u_1)$ and $h(3u_1) = 3h(u_1), \quad \forall\; u_1 \in J.$
For each positive integer a, it is generally true that
$h(au_1) = ah(u_1), \quad \forall\; u_1 \in J.$
Consequently we see that $h(\lambda u_1) = \lambda h(u_1)$, $\forall \lambda \in \mathbb{Z}$. Let $\lambda = \dfrac{m}{n}$, where $m \in \mathbb{Z}, n \in \mathbb{N}$. Hence $mu_1 = n(\lambda u_1)$ and by what has already been proved,
$h(mu_1) = mh(u_1) = mh(n(\lambda u_1)) = nh(\lambda u_1),$
which establishes the equality (iii), $\forall \lambda \in \mathbb{Q}$ □

Theorem 2.2. *A mapping* $h : J \to L$ *satisfies the functional equation (1.1),* $\forall u_1,\, u_2 \in J$ *if and only if* $h : J \to L$ *satisfies the functional equation (1.7),* $\forall u_1,\, u_2,\, u_3 \in\; J$.

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1, u_2 \in J$. Replacing u_2 by $u_2 + u_3$ in (1.1) and using Theorem 2.1, we obtain

$$h(u_1 + u_2 + u_3) = h(u_1) + h(u_2) + h(u_3).$$

Similarly, replacing u_2 by $u_2 - u_3$, $-u_2 + u_3$, and $-u_2 - u_3$ respectively in (1.1), we get the following equations:

$$\begin{aligned} h(u_1 + u_2 - u_3) &= h(u_1) + h(u_2) + h(-u_3) \\ h(u_1 - u_2 + u_3) &= h(u_1) + h(-u_2) + h(u_3) \\ h(u_1 - u_2 - u_3) &= h(u_1) + h(-u_2) + h(-u_3) \end{aligned}$$

Adding these equations while considering the oddness of h results in equation (1.7). Thus, if h satisfies (1.1), it also satisfies (1.7).

Conversely, let $h : J \to L$ satisfy the functional equation (1.7), $\forall u_1, u_2, u_3 \in J$. By substituting $(u_1, u_2, u_3) = (0, 0, 0)$ into (1.7), we obtain $h(0) = 0$. Furthermore, by substituting $(u_1, u_2, u_3) = (0, 0, u_1)$ into (1.7), we deduce

$$h(-u_1) = -h(u_1), \quad \forall u_1 \in J.$$

Therefore, h is an odd function. Finally, by setting $u_3 = 0$ in (1.7), we derive

$$h(u_1 + u_2) + h(u_1 - u_2) = 2h(u_1), \quad \forall u_1, u_2 \in J.$$

Interchanging u_1 and u_2 in the previous equation and utilizing the oddness of h, we obtain

$$h(u_1 + u_2) - h(u_1 - u_2) = 2h(u_2), \quad \forall u_1, u_2 \in J.$$

Adding the last two equations together, we arrive at equation (1.1). □

Theorem 2.3. *A mapping* $h : J \to L$ *satisfies the functional equation (1.1) ,* $\forall u_1,\, u_2 \in J$ *if and only if* $h : J \to L$ *satisfies the functional equation (1.8),* $\forall u_1,\, u_2,\, u_3 \in\; J$.

Proof. Let $h : J \to L$ satisfy the functional equation (1.1), $\forall u_1, u_2 \in J$. Replacing (u_1, u_2) by $(3u_1, 2u_2 + u_3)$ in (1.1) and using Theorem 2.1, we obtain

$$h(3u_1 + 2u_2 + u_3) = 3h(u_1) + 2h(u_2) + h(u_3).$$

Similarly, replacing u_2 by $u_2 - u_3$, $-u_2 + u_3$, and $-u_2 - u_3$ respectively in (1.1), we get the following equations:

$$h(3u_1 + 2u_2 - u_3) = 3h(u_1) + 2h(u_2) + h(-u_3)$$
$$h(3u_1 - 2u_2 + u_3) = 3h(u_1) + 2h(-u_2) + h(u_3)$$
$$h(3u_1 - 2u_2 - u_3) = 3h(u_1) + 2h(-u_2) + h(-u_3)$$

Adding these equations while considering the oddness of h results in equation (1.8). Thus, if h satisfies (1.1), it also satisfies (1.8).

Conversely, let $h : J \to L$ satisfy the functional equation (1.8), $\forall u_1, u_2, u_3 \in J$. By substituting $(u_1, u_2, u_3) = (0, 0, 0)$ into (1.8), we obtain $h(0) = 0$. Furthermore, by substituting $(u_1, u_2, u_3) = (0, 0, u_1)$ into (1.8), we deduce

$$h(-u_1) = -h(u_1), \quad \forall u_1 \in J.$$

Therefore, h is an odd function. By setting $(u_1, u_2, u_3) = (u_1, 0, 0)$ in (1.8), we derive

$$h(3u_1) = 3h(u_1), \quad \forall u_1, u_2 \in J.$$

Again setting u_1 into $\frac{u_1}{3}$ in the previous equation, we obtain

$$h\left(\frac{u_1}{3}\right) = \frac{1}{3}h(u_1), \quad \forall u_1 \in J.$$

By setting $(u_1, u_2, u_3) = \left(\frac{u_1}{3}, \frac{u_2}{2}, 0\right)$ in (1.8) and using the above equation, we derive

$$h(u_1 + u_2) + h(u_1 - u_2) = 2h(u_1), \quad \forall u_1, u_2 \in J.$$

The remaining argument is quite similar to that presented for Theorem 2.2. □

We can establish the following theorem in a manner analogous to the proof of the preceding two theorems. For the sake of brevity, we shall skip the proof.

Theorem 2.4. *A mapping $h : J \to L$ satisfies the functional equation (1.1) $\forall u_1,\ u_2 \in J$ if and only if $h : J \to L$ satisfies the functional equation (1.9) $\forall u_1,\ u_2,\ u_3 \in\ J$.*

Theorems 2.2, 2.3, and 2.4 establish that a mapping $h : J \to L$ satisfies the functional equation (1.1) $\forall u_1,\ u_2,\ u_3 \in\ J$ if and only if it satisfies the functional equations (1.7), (1.8), and (1.9) $\forall u_1,\ u_2,\ u_3 \in\ J$, respectively.

Based on the aforementioned theorems, we can state the following theorem without presenting the proof:

Theorem 2.5. *Let $h : J \to L$ be a mapping. Then the following are equivalent:*

(i) *h satisfies the functional equation (1.1), $\forall u_1,\ u_2,\ u_3 \in\ J$.*

(ii) *h satisfies the functional equation (1.7), $\forall u_1,\ u_2,\ u_3 \in\ J$.*

(iii) *h satisfies the functional equation (1.8), $\forall u_1,\ u_2,\ u_3 \in\ J$.*

(iv) *h satisfies the functional equation (1.9), $\forall u_1,\ u_2,\ u_3 \in\ J$.*

In this article, hereafter we will use the following notation: Let J represents a normed space, and L and K represent Banach spaces. We define a mapping $Dh : J^3 \to L$ as follows:

$$\begin{aligned} Dh(u_1, u_2, u_3) =& h(3u_1 + 2u_2 + u_3) + h(3u_1 + 2u_2 - u_3) + h(3u_1 - 2u_2 + u_3) \\ &+ h(3u_1 - 2u_2 - u_3) - 12h(u_1), \quad \forall u_1, u_2, u_3 \in J. \end{aligned} \tag{2.1}$$

3 Stability Results

The stability of the additive functional equation (1.8) is demonstrated under generalized Hyers-Ulam conditions.

Theorem 3.1. *Let $j \in \{-1, 1\}$ and consider a function $\psi : J^3 \to [0, \infty)$ satisfying the following condition:*

$$\lim_{n\to\infty} \frac{\psi\left(6^{nj}u_1,\, 6^{nj}u_2,\, 6^{nj}u_3\right)}{6^{nj}} = 0, \quad \forall u_1,\, u_2,\, u_3 \in J. \tag{3.1}$$

Next, let $h : J \to L$ be a function that meets the inequality:

$$\|Dh(u_1,\, u_2,\, u_3)\| \le \psi\left(u_1,\, u_2,\, u_3\right), \quad \forall u_1,\, u_2,\, u_3 \in J. \tag{3.2}$$

Under these conditions, exists a unique additive mapping $A : J \to L$ that meets (1.8) and the following stability condition:

$$\|h(u_1) - A(u_1)\| \le \frac{1}{6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(6^{kj}u_1\right)}{6^{kj}}, \quad \forall\ u_1 \in J, \tag{3.3}$$

Here, $\xi\left(u_1\right)$ and $A(u_1)$ are defined as follows: $\xi\left(u_1\right) = \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right)$, and
$A(u_1) = \lim_{n\to\infty} \frac{h(6^{nj}u_1)}{6^{nj}}, \quad \forall\ u_1 \in J.$

Proof. By substituting $(u_1,\, u_2,\, u_3)$ with $(u_1,\, u_1,\, u_1)$ in (3.2), we arrive at the inequality:

$$\|h(6u_1) + h(2u_1) - 12h(u_1)\| \le \psi\left(u_1,\, u_1,\, u_1\right), \quad \forall\ u_1 \in J. \tag{3.4}$$

Similarly, by replacing $(u_1,\, u_2,\, u_3)$ with $(u_1,\, 0,\, u_1)$ in (3.2), we obtain:

$$\|h(4u_1) + h(2u_1) - 6h(u_1)\| \le \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J. \tag{3.5}$$

Combining (3.4) and (3.5), we can deduce that:

$$\begin{aligned}
&\|h(6u_1) - 6h(u_1)\| \\
&= \|h(6u_1) + h(4u_1) + h(2u_1) - 12h(u_1) - h(4u_1) - h(2u_1) + 6h(u_1)\| \\
&\le \|h(6u_1) + h(2u_1) - 12h(u_1)\| + \|h(4u_1) + h(2u_1) - 6h(u_1)\| \\
&\le \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J.
\end{aligned} \tag{3.6}$$

Dividing this inequality by 6, we have:

$$\left\| \frac{h(6u_1)}{6} - h(u_1) \right\| \le \frac{\xi(u_1)}{6} \tag{3.7}$$

where $\xi(u_1) = \psi\left(u_1,\, u_1,\, u_1\right) + \frac{1}{2}\psi\left(u_1,\, 0,\, u_1\right), \quad \forall\ u_1 \in J$. Continuing, by replacing u_1 with $6u_1$ and dividing by 6 in (3.7), we obtain:

$$\left\| \frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6} \right\| \le \frac{\xi(6u_1)}{6^2}, \quad \forall\ u_1 \in J. \tag{3.8}$$

From (3.7) and (3.8), it follows that:

$$\begin{aligned}
\left\| \frac{h(6^2u_1)}{6^2} - h(u_1) \right\| &\le \left\| \frac{h(6u_1)}{6} - h(u_1) \right\| + \left\| \frac{h(6^2u_1)}{6^2} - \frac{h(6u_1)}{6} \right\| \\
&\le \frac{1}{6}\left[\xi(u_1) + \frac{\xi(6u_1)}{6}\right], \quad \forall\ u_1 \in J.
\end{aligned} \tag{3.9}$$

Mathematical induction on a positive integer n allows us to continue the investigation and yields:

$$\left\| \frac{h(6^n u_1)}{6^n} - h(u_1) \right\| \leq \frac{1}{6} \sum_{k=0}^{n-1} \frac{\xi(6^k u_1)}{6^k} \leq \frac{1}{6} \sum_{k=0}^{\infty} \frac{\xi(6^k u_1)}{6^k}, \quad \forall\, u_1 \in J. \tag{3.10}$$

Replacing u_1 with $6^m u_1$ and dividing by 6^m in (3.10) shows the convergent behavior of the sequence $\left\{ \frac{h(6^n u_1)}{6^n} \right\}$; for every $m, n > 0$, we have:

$$\begin{aligned}
\left\| \frac{h(6^{n+m} u_1)}{6^{(n+m)}} - \frac{h(6^m u_1)}{6^m} \right\| &= \frac{1}{6^m} \left\| \frac{h(6^n \cdot 6^m u_1)}{6^n} - h(6^m u_1) \right\| \\
&\leq \frac{1}{6} \sum_{k=0}^{n-1} \frac{\xi(6^{k+m} u_1)}{6^{(k+m)}} \\
&\leq \frac{1}{6} \sum_{k=0}^{\infty} \frac{\xi(6^{k+m} u_1)}{6^{(k+m)}} \\
&\to 0 \quad as \;\; m \to \infty, \quad \forall\, u_1 \in J.
\end{aligned}$$

Hence $\left\{ \frac{h(6^n u_1)}{6^n} \right\}$ is Cauchy. Since L is complete, there exists a mapping $A : J \to L$ such that: $A(u_1) = \lim_{n\to\infty} \frac{h(6^n u_1)}{6^n}, \quad \forall\, u_1 \in J$. We can show that (3.3) holds $\forall u_1 \in J$ when we extend n in (3.10) to infinity.

To prove that A meets (1.8), we can replace (u_1, u_2, u_3) by $(6^n u_1, 6^n u_2, 6^n u_3)$ and divide by 6^n in (**??**, leading to:

$$\begin{aligned}
\frac{1}{6^n} \Big\| h(6^n(3u_1 + 2u_2 + u_3)) + h(6^n(3u_1 + 2u_2 - u_3)) + h(6^n(3u_1 - 2u_2 + u_3)) \\
+ h(6^n(3u_1 - 2u_2 - u_3)) - 12h(6^n u_1) \Big\| \\
\leq \frac{1}{6^n} \psi(6^n u_1, 6^n u_2, 6^n u_3), \quad \forall u_1, u_2, u_3 \in J.
\end{aligned}$$

Allowing n to tend towards infinity in the previous inequality and applying the characterization of $A(u_1)$, we obtain:

$$A(3u_1 + 2u_2 + u_3) + A(3u_1 + 2u_2 - u_3) + A(3u_1 - 2u_2 + u_3) + A(3u_1 - 2u_2 - u_3) - 12A(u_1).$$

Hence A meets (1.8) $\forall u_1, u_2, u_3 \in J$. To prove that A is unique, if $B(u_1)$ is another additive mapping satisfying (1.8) and (3.3), then:

$$\begin{aligned}
\|A(u_1) - B(u_1)\| &= \frac{1}{6^n} \|A(6^n u_1) - B(6^n u_1)\| \\
&\leq \frac{1}{6} 6^n \left\{ \|A(6^n u_1) - h(6^n u_1)\| + \|h(6^n u_1) - B(6^n u_1)\| \right\} \\
&\leq \frac{2}{6} \sum_{k=0}^{\infty} \frac{\xi(6^{k+n} u_1)}{6^{(k+n)}} \\
&\to 0 \quad as \;\; n \to \infty, \quad \forall\, u_1 \in J.
\end{aligned}$$

Hence A is unique. Therefore, the theorem holds for both $j = 1$ and $j = -1$. The proof of the theorem is now complete. □

Corollary 3.2. *Let λ and s be non-negative real numbers. Consider a function $h : J \to L$ that meets the following inequality:*

$$\|Dh(u_1, u_2, u_3)\| \leq \begin{cases} \lambda, & \\ \lambda \|u_1\|^s + \|u_2\|^s + \|u_3\|^s, & \text{if } s \neq 1; \\ \lambda \|u_1\|^s \|u_2\|^s \|u_3\|^s, & \text{if } 3s \neq 1; \\ \lambda \|u_1\|^s \|u_2\|^s \|u_3\|^s + \|u_1\|^{3s} + \|u_2\|^{3s} + \|u_3\|^{3s}, & \text{if } 3s \neq 1; \end{cases} \tag{3.11}$$

for all $u_1, u_2, u_3 \in J$. Under these conditions, there exists a unique additive function $A : J \to L$ such that the following stability condition holds:

$$\|h(u_1) - A(u_1)\| \leq \begin{cases} \frac{3\lambda}{10}, & \\ \frac{4\lambda \|u_1\|^s}{|6-6^s|}, & \text{if } s \neq 1; \\ \frac{\lambda \|u_1\|^{3s}}{|6-6^{3s}|}, & \text{if } 3s \neq 1; \\ \frac{5\lambda \|u_1\|^{3s}}{|6-6^{3s}|}, & \text{if } 3s \neq 1; \end{cases} \tag{3.12}$$

for all $u_1 \in J$.

4 Conclusions

In conclusion, this research article has employed a direct method to ascertain the solution of the specified equation and to explore its generalized Hyers-Ulam stability within the context of Banach spaces.

Through our findings in Section 2, we have demonstrated the existence of general solutions to Equations (1.7), (1.8), and (1.9). Furthermore, in Section 3, we have utilized the Hyers direct approach to assess the stability of the functional equation (1.8) under the generalized Hyers-Ulam conditions. These results contribute to a deeper understanding of the stability properties of the given equation and provide valuable insights into its solutions in the realm of Banach spaces.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Ulam, SM. Problems in Modern Mathematics. Science Editions, Wiley, New York; 1964.

[2] Hyers, DH. On the stability of the linear functional equation. Proc. Nat. Acad. Sci., U.S.A. 1941;27:222-224.

[3] Aoki, T. On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan. 1950;2:64-66.

[4] Rassias, ThM. On the stability of the linear mapping in Banach spaces. Proc. Amer. Math Soc. 1978;72:297-300.

[5] Aczel, J., Dhombres, J. Functional Equations in Several Variables. Cambridge Univ. Press; 1989.

[6] Gajda, Z. On stability of additive mappings. Int. J. Math. Sci. 1991;14:431-434.

[7] Rassias, JM. On approximately of approximately linear mappings by linear mappings. J. Funct. Anal. USA. 1982;46:126-130.

[8] Rassias, JM. Solution of a problem of Ulam. J. Approx. Theory. 1989;57:268-273.

[9] Rassias, JM. Solution of a stability problem of Ulam. Discuss. Math. 1992;12:95-103.

[10] Rassias, JM. On the stability of the Euler-Lagrange functional equation. Chin. J. Math. 1992;20:185-190.

[11] Găvruţă, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. 1994;184:431-436.

[12] Arunkumar, M. Solution and stability of Arun-additive functional equations. Int. J. Math. Sci. Eng. Appl. 2010;4(3):33-46.

[13] Czerwik, S. Functional Equations and Inequalities in Several Variables. World Scientific, River Edge, NJ. 2002.

[14] Eskandani, GZ., Găvruţă, P. On the stability problem in quasi-Banach spaces. Nonlinear Funct. Anal. Appl. 2012;5:459-465.

[15] Govindan, V., Alb Lupaş, A., Noeiaghdam, S. Stability of Additive Functional Equation Originating from Characteristic Polynomial of Degree Three. Symmetry. 2022;14(4):700.

[16] Hyers, DH., Isac, G., Rassias, Th.M. Stability of functional equations in several variables. Birkhauser, Basel; 1998.

[17] Jung, SM. Hyers-Ulam-Rassias Stability for Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor; 2001.

[18] Rassias, JM, Rassias, MJ. Asymptotic behavior of alternative Jensen and Jensen type functional equations. Bull. Sci. Math. 2005;129:545-558.

[19] Rassias, JM. Refined Hyers Ulam approximation of approximately Jensen type mappings. Bull. Sci. Math. 2007;131:89-98.

[20] Rassias, JM, Kim, HM. Generalized Hyers-Ulam stability for general additive functional equations in quasi$-\beta-$normed spaces. J. Math. Anal. Appl. 2009;356(1):302-309.

[21] Tober, J. Stability of Cauchy functional equation in quasi-Banach spaces. Ann. Polon. Math. 2004;83:243-255.

[22] Arunkumar, M. Three-dimensional Quartic Functional Equation in Fuzzy Normed spaces. Far East J. Appl. Math. 2010;41(2):83–94.

[23] Lee, DO. Hyers-Ulam stability of an additive type functional equation. J. Appl. Math. Comput. 2003;13(1-2):471-477.

[24] Narasimmon, P. Solution and stability of a generalized k–additive functional equation. Journal of Interdisciplinary Mathematics. 2018;21(1):171-184.

[25] Agilan, P, Rassias, JM, Vallinayagam, V. Hyers Stability of Additive Functional Equation in Banach Space. In: Mohapatra, RN, Yugesh, S, Kalpana, G, Kalaivani, C. (eds) Mathematical Analysis and Computing. ICMAC 2019. Springer Proceedings in Mathematics & Statistics, vol 344. Springer, Singapore. 2021.

[26] Agilan, P, Julietraja, K, Almazah, MMA, Alsinai, A. Stability Analysis of a New Class of Series Type Additive Functional Equation in Banach Spaces: Direct and Fixed Point Techniques. Mathematics. 2023;11(4):887.

[27] Park, C, Rassias, MT. Additive functional equations and partial multipliers in algebras. RACSAM. 2019;113:2261–2275.

[28] Pasupathi, A, Konsalraj, J, Fatima, N, Velusamy, V, Mlaiki, N, Souayah, N. Direct and Fixed-Point Stability–Instability of Additive Functional Equation in Banach and Quasi-Beta Normed Spaces. Symmetry. 2022;14(8):1700.

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