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The Rectangle Rule for Computing Cauchy Principal Value Integral on Circle

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Abstract

The classical composite rectangle (constant) rule for the computation of Cauchy principle value integral with the singular kernel $\cot\frac{x-s}{2}$ is discussed. We show that the superconvergence rate of the composite midpoint rule occurs at certain local coodinate of each subinterval and obtain the corresponding superconvergence error estimate. Then collation methods are presented to solve certain kind of Hilbert singular integral equation. At last, some numerical examples are provided to validate the theoretical analysis.

Keywords

Cauchy Principal Value Integral, Extrapolation Method, Composite Rectangle Rule, Superconvergence, Error Expansion

1. Introduction

Consider the Cauchy principle integral

$$I(f;s)=\fint_{c}^{c+2\pi}\cot\frac{x-s}{2}f(x)\mathrm{d}x=g(s),s\in(0,2\pi) \tag{1}$$

where $\fint_{c}^{c+2\pi}$ denotes a Cauchy principle value integral and s is the singular point.

There are several different definitions which can be proved equally, such as the definition of subtraction of the singularity, regularity definition, direct definition and so on. In this paper we adopt the following one

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$$\int\!\!\!\!\!\!-_{c}^{c+2\pi} \cot\frac{x-s}{2} f(x)\mathrm{d}x = \lim_{\varepsilon\to 0}\left\{\int_0^{s-\varepsilon}\cot\frac{x-s}{2} f(x)\mathrm{d}x + \int_{s+\varepsilon}^{2\pi}\cot\frac{x-s}{2} f(x)\mathrm{d}x\right\}, \tag{2}$$

Cauchy principal value integrals have recently attracted a lot of attention [1]-[5]. The main reason for this interest is probably due to the fact that integral equations with Cauchy principal value integrals have shown to be an adequate tool for the modeling of many physical situations, such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. It is the aim of this paper to investigate the superconvergence phenomenon of rectangle rule for it and, in particular, to derive error estimates.

The superconvergence of composite Newton-Cotes rules for Hadamard finite-part integrals was studied in [6]-[8], where the superconvergence rate and the superconvergence point were presented, respectively. Lyness [9] derived the Euler-Maclaurin formula for Cauchy principal value integrals. Elliott and Venturino [2] employed sigmoidal transformations to obtain better approximation to Cauchy principal value integrals. In the reference Avram Sidi [10] [11] and [12] presented high-accuracy numerical quadrature methods for integrals of singular periodic functions. The classical Euler-Maclaurin summation formula [13] expressed the difference between a definite integral over $[0,1]$ and its approximation using the trapezoidal rule with step length $h=1/m$ as an asymptotic expansion in powers of h together with a remainder term.

The extrapolation method for the computation of Hadamard finite-part integrals on the interval and in a circle is studied in [14] and [15] which focus on the asymptotic expansion of error function. Based on the asymptotic expansion of the error functional, algorithm with theoretical analysis of the generalized extrapolation is given.

In this paper, the density function $f(x)$ is replaced by the approximation function $f_C(x)$ while the singular kernel $\cot\frac{x-s}{2}$ is computed analysis in each subinterval, where $f_C(x)$ is the midpoint rectangle rule. This methods may be considered as the semi-discrete methods and the order of singularity kernel can be reduced somehow. This idea was firstly presented by Linz [16] in the paper to calculated the hypersingular integral on interval. He used the trapezoidal rule and Simpson rule to approximate the density function $f(x)$ and the convergence rate was $O(h^k), k=1,2$ when the singular point was always located at the middle of certain subinterval. This paper focuses on the superconvergence of mid-rectangle rule for Cauchy principle integrals. We prove both theoretically and numerically that the composite mid-rectangle rule reaches the superconvergence rate when the local coordinate of the singular point s is ± 1. Then a collation methods is presented to solve certain kind of Hilbert singular integral equation.

The rest of this paper is organized as follows. In Sect. 2, after introducing some basic formulas of the rectangle rule, we present the main resluts. In Sect. 3, we perform the proof. Finally, several numerical examples are provided to validate our analysis.

2. Main Result

Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c+2\pi$ be a uniform partition of the interval $[c, c+2\pi]$ with mesh size $h = 2\pi/n$. Define by $f_C(x)$ the piecewise constant interpolant for $f(x)$

$$f_C(x) = f(\hat{x}_j),\ \hat{x}_j = x_{j-1} + \frac{h}{2}, j = 1,2,\cdots,n \tag{3}$$

and a linear transformation

$$x = \hat{x}_j(\tau) := (\tau+1)(x_{j+1}-x_j)/2 + x_j,\ \tau\in[-1,1], \tag{4}$$

from the reference element $[-1,1]$ to the subinterval $[x_j, x_{j+1}]$. Replacing $f(x)$ in (2) with $f_C(x)$ gives the composite rectangle rule:

$$I_n(f;s) := \int\!\!\!\!\!\!-_{c}^{c+2\pi}\cot\frac{x-s}{2} f(x)\mathrm{d}x = \sum_{i=1}^{n}\omega_j(s) f(\hat{x}_j) = I(f;s) - E_n(f,s), \tag{5}$$

where $E_n(f,s)$ denotes the error functional and $\omega_i(s)$ is the Cote coefficients given by

$$\omega_j(s) = h\cot\frac{\hat{x}_j - s}{2}. \tag{6}$$

We also define

$$k_s(x)=\begin{cases}(x-s)\cot\dfrac{x-s}{2}, & x\neq s,\\ 2, & x=s.\end{cases} \tag{7}$$

Theorem 1: *Assume* $f(x)\in C^1[c,c+2\pi]$. *For the rectangle rule* $I_n(f,s)$ *defined as* (5). *Assume that* $s=x_m+(1+\tau)h/2$, *there exist a positive constant C, independent of h and s, such that*

$$\left|E_n(f,s)\right|\le C\left(\left|\ln h\right|+\left|\ln\gamma(\tau)\right|\right)h, \tag{8}$$

where

$$\gamma(\tau)=\min_{0\le j\le n}\frac{\left|s-x_j\right|}{h}=\frac{1-|\tau|}{2}. \tag{9}$$

Proof: Let $R(x)=f(x)-f_C(x)$, then we have $\left|R(x)\right|\le Ch$. As

$$\begin{aligned}E_n(f,s)&=⨍_c^{c+2\pi}\cot\frac{x-s}{2}R(x)\mathrm{d}x\\&=2⨍_c^{c+2\pi}(x-s)\cot\frac{x-s}{2}\frac{R(x)}{x-s}\mathrm{d}x\\&=2⨍_c^{c+2\pi}\frac{R(x)}{x-s}\mathrm{d}x+⨍_c^{c+2\pi}\frac{k_s(x)-2}{x-s}R(x)\mathrm{d}x\end{aligned} \tag{10}$$

For the first part of (10), we have

$$\begin{aligned}\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{R(x)}{x-s}\mathrm{d}x\right|&\le Ch\left(\int_c^{x_m}\frac{1}{s-x}\mathrm{d}x+\int_{x_{m+1}}^{c+2\pi}\frac{1}{x-s}\mathrm{d}x\right)\\&=Ch\ln\frac{(c+2\pi-s)(s-c)}{(x_{m+1}-s)(s-x_m)}\\&\le C\left(\left|\ln h\right|+\left|\ln\gamma(\tau)\right|\right)h\end{aligned} \tag{11}$$

For the second part of (10),

$$\left|⨍_{x_m}^{x_{m+1}}\frac{R(x)}{x-s}\mathrm{d}x\right|\le\left|\int_{x_m}^{x_{m+1}}\frac{R(x)-R(s)}{x-s}\right|+\left|R(s)\ln\frac{x_{m+1}-s}{s-x_m}\right|\le Ch\left|\ln\gamma(\tau)\right| \tag{12}$$

$$\left|⨍_c^{c+2\pi}\frac{k_s(x)-2}{x-s}R(x)\mathrm{d}x\right|\le Ch\left|⨍_c^{c+2\pi}\frac{k_s(x)-2}{x-s}\mathrm{d}x\right|\le Ch\left|\ln\gamma(\tau)\right|. \tag{13}$$

Combining (11) and (13) together, the proof is completed.

Setting

$$I_{n,j}(s)=\begin{cases}⨍_{x_m}^{x_{m+1}}\left[\cot\dfrac{x-s}{2}-\cot\dfrac{\hat{x}_m-s}{2}\right]\mathrm{d}x, & j=m,\\ \displaystyle\int_{x_i}^{x_{j+1}}\left[\cot\dfrac{x-s}{2}-\cot\dfrac{\hat{x}_j-s}{2}\right]\mathrm{d}x, & j\neq m.\end{cases} \tag{14}$$

Lemma 1: *Assume* $s=x_m+(\tau+1)h/2$ *with* $\tau\in[-1,1]$. *Let* $I_{n,j}(s)$ *be defined by* (14), *then there holds that*

$$I_{n,j}(s)=\sum_{k=1}^{\infty}\frac{1}{k}\left(\cos k\left(x_{j+1}-s\right)-\cos k\left(x_j-s\right)\right)-h\sum_{k=1}^{\infty}\sin k\left(\hat{x}_{j+1}-s\right) \tag{15}$$

Proof: For $i=m$, by the definition of cauchy principal value integral, we have

$$I_{n,m}(s)=\lim_{\varepsilon\to 0}\left(\int_{x_m}^{s-\varepsilon}+\int_{s+\varepsilon}^{x_{m+1}}\right)\left[\cot\frac{x-s}{2}-\cot\frac{\hat{x}_m-s}{2}\right]\mathrm{d}x$$
$$=2\ln\left(2\sin\frac{x_m-s}{2}\right)-2\ln\left(2\sin\frac{x_{m+1}-s}{2}\right)-h\left[\cot\frac{\hat{x}_m-s}{2}\right] \tag{16}$$

For $i\neq m$, taking integration by parts on the correspondent Riemann integral, we have

$$I_{n,i}(s)=2\ln\left(2\sin\frac{x_i-s}{2}\right)-2\ln\left(2\sin\frac{x_{i+1}-s}{2}\right)-h\left[\cot\frac{\hat{x}_i-s}{2}\right] \tag{17}$$

Now, by using the well-known identity

$$\ln\left|2\sin\frac{x}{2}\right|=-\sum_{n=1}^{\infty}\frac{1}{n}\cos nx \tag{18}$$

and

$$\frac{1}{2}\cot\frac{x}{2}=\sum_{n=1}^{\infty}\sin nx \tag{19}$$

The proof is completed.

By the identity in [17]

$$\pi\cot\pi x=\sum_{l=-\infty}^{l=\infty}\frac{1}{x+l}, \tag{20}$$

then we get

$$\cot\frac{x-s}{2}=\frac{2}{x-s}+\sum_{l=1}^{\infty}\frac{2}{x-s-2l\pi}+\sum_{l=1}^{\infty}\frac{2}{x-s+2l\pi} \tag{21}$$

and

$$\cot\frac{x-s}{2}-\cot\frac{\hat{x}_m-s}{2}=\frac{2}{x-s}+\sum_{l=1}^{\infty}\frac{2}{x-s-2l\pi}+\sum_{l=1}^{\infty}\frac{2}{x-s+2l\pi}$$
$$-\left[\frac{2}{\hat{x}_m-s}+\sum_{l=1}^{\infty}\frac{2}{\hat{x}_m-s-2l\pi}+\sum_{l=1}^{\infty}\frac{2}{\hat{x}_m-s+2l\pi}\right]$$
$$=\frac{2(\hat{x}_m-x)}{(x-s)(\hat{x}_m-s)}+\sum_{l=1}^{\infty}\frac{2(\hat{x}_m-x)}{(x-s-2l\pi)(\hat{x}_m-s-2l\pi)}$$
$$+\sum_{l=1}^{\infty}\frac{2(\hat{x}_m-x)}{(x-s+2l\pi)(\hat{x}_m-s+2l\pi)} \tag{22}$$

For $j=m$, by the definition of cauchy principal value integral, we have

$$I_{n,m}(s)=\lim_{\varepsilon\to 0}\left(\int_{x_m}^{s-\varepsilon}+\int_{s+\varepsilon}^{x_{m+1}}\right)\left[\cot\frac{x-s}{2}-\cot\frac{\hat{x}_m-s}{2}\right]\mathrm{d}x$$
$$=\lim_{\varepsilon\to 0}\left(\int_{x_m}^{s-\varepsilon}+\int_{s+\varepsilon}^{x_{m+1}}\right)\frac{2(\hat{x}_m-x)}{(x-s)(\hat{x}_m-s)}\mathrm{d}x$$
$$+\sum_{l=1}^{\infty}\int_{x_m}^{x_{m+1}}\frac{2(\hat{x}_m-x)}{(x-s-2l\pi)(\hat{x}_m-s-2l\pi)}\mathrm{d}x$$
$$+\sum_{l=1}^{\infty}\int_{x_m}^{x_{m+1}}\frac{2(\hat{x}_m-x)}{(x-s+2l\pi)(\hat{x}_m-s+2l\pi)}\mathrm{d}x. \tag{23}$$

Let $Q_n(x)$ be the function of the second kind associated with the Legendre polynomial $P_n(x)$, defined by (cf. [17])

$$Q_0(x)=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|,\quad Q_1(x)=xQ_0(x)-1. \tag{24}$$

We also define

$$W(f,\tau):=f(\tau)+\sum_{i=0}^{\infty}\left[f(2i+\tau)+f(-2i+\tau)\right],\tau\in(-1,1). \tag{25}$$

Then, by the definition of W,

$$\begin{aligned}W(Q_0)(\tau)&=\frac{1}{2}\ln\frac{1+\tau}{1-\tau}+\frac{1}{2}\sum_{i=1}^{\infty}\left(\ln\frac{2i+1+\tau}{2i-1+\tau}+\ln\frac{2i-1-\tau}{2i+1-\tau}\right)\\&=\frac{1}{2}\lim_{i\to\infty}\ln\frac{2i+1+\tau}{2i+1-\tau}=0,\\W(xQ_0')(\tau)&=\frac{\tau}{1-\tau^2}-\sum_{i=1}^{\infty}\left(\frac{2i+\tau}{(2i+\tau)^2-1}+\frac{-2i+\tau}{(-2i+\tau)^2-1}\right)\\&=\frac{1}{2}\lim_{n\to\infty}\sum_{k=-n}^{k=n}\frac{1}{k+\frac{1}{2}+\frac{\tau}{2}}=-\frac{\pi}{2}\tan\frac{\pi(1+\tau)}{2},\end{aligned}$$

it follows that

$$W(Q_0+xQ_0',\tau)=-\pi\tan\frac{\pi(\tau+1)}{2}. \tag{26}$$

Theorem 2: *Assume* $f(x)\in C^{2l}[c,c+2\pi]$. *For the rectangle rule* $I_n(f,s)$ *defined as* (5). *Assume that* $s=x_m+(1+\tau)h/2$, *there exist a positive constant C, independent of h and s, such that*

$$E_n(f,s)=-f(s)\pi\tan\frac{\pi(\tau+1)}{2}+\mathcal{R}_n(s), \tag{27}$$

where

$$|\mathcal{R}_n(s)|\le C\max\{|k_s(x)|\}(|\ln h|+|\ln\gamma(\tau)|)h^{2l} \tag{28}$$

$\gamma(\tau)$ is defined as (9).

It is known that the global convergence rate of the composite rectangle rule is lower than Riemann integral.

3. Proof of the Theorem

In this section, we study the superconvergence of the composite rectangle rule for Cauchy principle integrals.

Preliminaries

In the following analysis, C will denote a generic constant that is independent of h and s and it may have different values in different places.

Lemma 2: *Under the same assumptions of Theorem* 2, *it holds that*

$$\begin{aligned}&\cot\frac{x-s}{2}f(x)-\cot\frac{\hat{x}_j-s}{2}f_C(x)\\&=\left[\cot\frac{x-s}{2}-\cot\frac{\hat{x}_j-s}{2}\right]f(s)\\&\quad+\sum_{i=1}^{2l-1}\frac{f^{(i)}(s)}{i!}\left[(x-s)^i\cot\frac{x-s}{2}-(\hat{x}_j-s)^i\cot\frac{\hat{x}_j-s}{2}\right]\\&\quad+\frac{f^{(2l)}(s_1)}{(2l)!}(x-s)^{2l}\cot\frac{x-s}{2}-\frac{f^{(2l)}(s_2)}{(2l)!}(\hat{x}_j-s)^{2l}\cot\frac{\hat{x}_j-s}{2}.\end{aligned}$$

where $s_1 \in \left(\hat{x}_j, s\right), s_2 \in \left(x, s\right)$.

Proof: Performing Taylor expansion of $f_C\left(x\right)$ at the point *x*, we have

$$f_C\left(\hat{x}_j\right) = f\left(s\right) + \sum_{i=1}^{2l-1} \frac{f^{(i)}\left(s\right)}{i!}\left(\hat{x}_j - s\right)^i + \frac{f^{(2l)}\left(s_1\right)}{(2l)!}\left(\hat{x}_j - s\right)^{2l} \cot\frac{\hat{x}_j - s}{2} \tag{29}$$

and

$$f\left(x\right) = f\left(s\right) + \sum_{i=1}^{2l-1} \frac{f^{(i)}\left(s\right)}{i!}\left(x - s\right)^i + \frac{f^{(2l)}\left(s_2\right)}{(2l)!}\left(x - s\right)^{2l} \cot\frac{x - s}{2}. \tag{30}$$

Combining (29) and (30) together we get the results.

Proof of Theorem 2: we have

$$\begin{aligned}
&\left(\int_c^{x_m} + \int_{x_{m+1}}^{c+2\pi}\right)\cot\frac{x-s}{2} f\left(x\right)\mathrm{d}x - \sum_{j=0, j\neq m}^{n-1} h\cot\frac{x_j - s}{2} f\left(x_j\right) \\
&= \sum_{j=0, j\neq m}^{n-1} \int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2} f\left(x\right) - \cot\frac{\hat{x}_j - s}{2} f\left(\hat{x}_j\right)\right]\mathrm{d}x \\
&= \sum_{j=0, j\neq m}^{n-1} \int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2} - \cot\frac{\hat{x}_j - s}{2}\right] f\left(s\right)\mathrm{d}x \\
&\quad + \sum_{i=1}^{2l-1} \frac{f^{(i)}\left(s\right)}{i!} \sum_{j=0, j\neq m}^{n-1} \int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2}\left(x-s\right)^i - \cot\frac{\hat{x}_j - s}{2}\left(\hat{x}_j - s\right)^i\right]\mathrm{d}x \\
&\quad + \sum_{j=0, j\neq m}^{n-1} \int_{x_j}^{x_{j+1}} \left[\frac{f^{(2l)}\left(s_1\right)}{(2l)!}\left(x-s\right)^{2l}\cot\frac{x-s}{2} - \frac{f^{(2l)}\left(s_2\right)}{(2l)!}\left(\hat{x}_j - s\right)^{2l}\cot\frac{\hat{x}_j - s}{2}\right]\mathrm{d}x.
\end{aligned} \tag{31}$$

For $i = m$, we have

$$\begin{aligned}
&⨍_{x_m}^{x_{m+1}} \cot\frac{x-s}{2} f\left(x\right)\mathrm{d}x - h\cot\frac{\hat{x}_m - s}{2} f\left(\hat{x}_m\right) = ⨍_{x_m}^{x_{m+1}} \cot\frac{x-s}{2} f\left(x\right) - \cot\frac{\hat{x}_m - s}{2} f\left(\hat{x}_m\right)\mathrm{d}x \\
&= ⨍_{x_m}^{x_{m+1}} \left[\cot\frac{x-s}{2} - \cot\frac{\hat{x}_m - s}{2}\right] f\left(s\right)\mathrm{d}x + \sum_{i=1}^{2l-1}\int_{x_m}^{x_{m+1}} \left[\cot\frac{x-s}{2}\left(x-s\right)^i - \cot\frac{\hat{x}_m - s}{2}\left(\hat{x}_m - s\right)^i\right]\mathrm{d}x \\
&\quad + \int_{x_m}^{x_{m+1}} \left[\frac{f^{(2l)}\left(s_1\right)}{(2l)!}\left(x-s\right)^{2l}\cot\frac{x-s}{2} - \frac{f^{(2l)}\left(s_2\right)}{(2l)!}\left(\hat{x}_j - s\right)^{2l}\cot\frac{\hat{x}_j - s}{2}\right]\mathrm{d}x
\end{aligned} \tag{32}$$

Putting (31) and (32) together yields

$$\begin{aligned}
&⨍_c^{c+2\pi} \cot\frac{x-s}{2} f\left(x\right)\mathrm{d}x - \sum_{j=0}^{n-1} h\cot\frac{\hat{x}_j - s}{2} f\left(\hat{x}_j\right) \\
&= \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2} f\left(x\right) - \cot\frac{\hat{x}_j - s}{2} f\left(\hat{x}_j\right)\right]\mathrm{d}x \\
&= \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2} - \cot\frac{\hat{x}_j - s}{2}\right] f\left(s\right)\mathrm{d}x \\
&\quad + \sum_{i=1}^{\infty} \frac{f^{(i)}\left(s\right)}{i!} \sum_{j=0, j\neq m}^{n-1} \int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2}\left(x-s\right)^i - \cot\frac{\hat{x}_j - s}{2}\left(\hat{x}_j - s\right)^i\right]\mathrm{d}x \\
&= S_0\left(\tau\right) f\left(s\right) + \sum_{i=1}^{2l-1} \frac{f^{(i)}\left(s\right)}{i!} \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \left[\cot\frac{x-s}{2}\left(x-s\right)^i - \cot\frac{\hat{x}_j - s}{2}\left(\hat{x}_j - s\right)^i\right]\mathrm{d}x \\
&\quad + \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \left[\frac{f^{(2l)}\left(s_1\right)}{(2l)!}\left(x-s\right)^{2l}\cot\frac{x-s}{2} - \frac{f^{(2l)}\left(s_2\right)}{(2l)!}\left(\hat{x}_j - s\right)^{2l}\cot\frac{\hat{x}_j - s}{2}\right]\mathrm{d}x.
\end{aligned} \tag{33}$$

Here

$$S_0(\tau)=\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\left[\cot\frac{x-s}{2}-\cot\frac{\hat{x}_j-s}{2}\right]dx$$

with the linear transformation from $[t_{j-1},t_j]$ to the identity interval $[-1,1]$. As for the last part of

$$\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\left[\cot\frac{x-s}{2}(x-s)^i-\cot\frac{\hat{x}_j-s}{2}(\hat{x}_j-s)^i\right]dx$$

which can be considered as the error estimate of left rectangle rule for the definite integral $\int_a^b(t-s)^{i-1}dt, i\geq 2$. Obviously,by the Theorem, it can be expanded by the Euler-Maclaurin expansions and we have

$$E_n^i(f,h)=\int_a^b(t-s)^{i-1}dt+\sum_{k=1}^{\infty}\frac{B_k(\theta)}{k!}\left[(b-s)^{(k-1)}-(a-s)^{(k-1)}\right]h^k,\ \ k\leq i-1 \tag{34}$$

It is easy to see that there are not relation with the singular point s which can be written as

$$E_n^i(f,h)=\int_a^b(t-s)^{i-1}dt+\sum_{k=1}^{\infty}c_kh^k,\ \ k\leq i-1. \tag{35}$$

The proof is complete. □

We actually obtain the error expansion of the rectangle rule and moreover, get the explicit expression of the first order term. So it is easy for us to get the superconvergence point with $S_0(\tau)=0$, which means that $\tau=\pm 1$ is the superconvergence point in subinterval not near the end of the interval.

Based on the theorem 1, we present the modify rectangle rule

$$\tilde{I}_n(f;s)=I_n(f;s)-f(s)\pi\tan\frac{\pi(\tau+1)}{2}. \tag{36}$$

4. Numerical Example

In this section, computational results are reported.

Example 1: *We consider the Hilbert singular integral with* $f(x)=\cos x+\sin x$ $c=0$. $s=c+x_{[n/4]}+(\tau+1)h/2$ *with* $\tau=\pm 1$ *is the superconvergence point.*

From **Table 1** and **Table 2**, we know that the superconvergence point is ± 1 with the coordinate location of singular point equal zero, while for the local coordinate of singular point do not equal zero,it is not convergence in general which coincides with our analysis.

For the modify classical rectangle rule, from **Table 3** and **Table 4**, for the non-superconvergence point and the supersonvergence point, we all get the supercocergence phenomenon.

In this section, we consider the integral equation

$$\frac{1}{2\pi}\fint_0^{2\pi}f(x)\cot\frac{x-s}{2}dx=g(s),\ \ s\in(0,2\pi), \tag{37}$$

with the compatibility condition

$$\int_0^{2\pi}g(x)dx=0. \tag{38}$$

As in [5], under the condition of (38), there exists a unique solution for the integral Equation (37). In order to get a unique solution, we adopt the following condition

$$\int_0^{2\pi}f(x)dx=0. \tag{39}$$

By choosing the middle points $\hat{x}_k=x_{k-1}+h/2(k=1,2,\cdots,n)$, we get the composite rectangle rule $I_n(f;s)$ to approximate the Hilbert singular integral in (37), then the following linear system is obtained

$$\frac{1}{\pi}\sum_{m=1}^{n}\left[h\cot\frac{x_m-\hat{x}_k}{2}\right]f_m=g(\hat{x}_k), k=1,2,\cdots,n, \tag{40}$$

Table 1. An error estimate of the rectangle rule $s = x_{[n/4]} + (\tau + 1)h/2$.

	$\tau = 1$	$\tau = -1$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$
8	−1.7764e−015	0	6.6963e−001	1.7335e+000
16	−1.7764e−015	0	2.2690e+000	4.1887e+000
32	0	0	2.9882e+000	5.2932e+000
64	−5.3291e−015	2.6645e−015	3.3190e+000	5.8039e+000
128	−4.4409e−014	−4.3521e−014	3.4762e+000	6.0477e+000
256	1.3323e−014	−9.7700e−015	3.5526e+000	6.1665e+000

Table 2. An error estimate of the rectangle rule $s = x_0 + (\tau + 1)h/2$.

	$\tau = 1$	$\tau = -1$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$
8	−1.1102e−015	−6.2172e−015	5.0863e+000	8.7150e+000
16	−3.1086e−015	−7.1054e−015	4.6011e+000	7.8365e+000
32	−8.8818e−015	−4.3521e−014	4.1701e+000	7.1371e+000
64	−2.6645e−015	−1.7764e−014	3.9119e+000	6.7284e+000
128	−2.6645e−015	−3.8192e−014	3.7729e+000	6.5102e+000
256	2.9310e−014	1.0658e−013	3.7010e+000	6.3978e+000

Table 3. An error estimate of the modify rectangle rule $s = x_{[n/4]} + (\tau + 1)h/2$.

	$\tau = 1$	$\tau = -1$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$
8	−1.7764e−015	0	−1.7764e−015	3.5527e−015
16	−1.7764e−015	0	−5.3291e−015	7.1054e−015
32	0	0	−3.5527e−015	2.3093e−014
64	−5.3291e−015	2.6645e−015	1.0658e−014	3.0198e−014
128	−4.4409e−014	−4.3521e−014	−4.7962e−014	−1.3323e−013
256	1.3323e−014	−9.7700e−015	−6.0396e−014	2.8422e−014

Table 4. An error estimate of the modify rectangle rule $s = x_0 + (\tau + 1)h/2$.

	$\tau = 1$	$\tau = -1$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$
8	−1.1102e−015	−6.2172e−015	−8.8818e−016	0
16	−3.1086e−015	−7.1054e−015	−2.8866e−015	−3.5527e−015
32	−8.8818e−015	−4.3521e−014	−9.5479e−015	−1.5099e−014
64	−2.6645e−015	−1.7764e−014	−7.3275e−015	−3.6637e−015
128	−2.6645e−015	−3.8192e−014	4.4409e−015	−2.6645e−014
256	2.9310e−014	1.0658e−013	3.1974e−014	1.2212e−014

and written as the matrix expression as

$$A_n F_n^a = G_n^e, \tag{41}$$

where

$$\begin{aligned} &A_n = \left(a_{km}\right)_{n\times n}, \\ &a_{km} = \frac{1}{\pi}\left[h\cot\frac{x_j - \hat{x}_k}{2}\right], k, m = 1, 2, \cdots, n, \\ &F_n^a = \left(f_1, f_2, \cdots, f_n\right)^{\mathrm{T}}, G_n^e = \left(g\left(\hat{x}_1\right), g\left(\hat{x}_2\right), \cdots, g\left(\hat{x}_n\right)\right)^{\mathrm{T}}, \end{aligned} \tag{42}$$

here $f_k\left(k = 1, 2, \cdots, n\right)$ denotes the numerical solution of f at $\hat{x}_k$. By directly calculation, we get A_n that is not only a symmetric Toeplitz matrix but also a circulant matrix. As for any $k = 1, 2, \cdots, n$,

$$\sum_{m=1}^{n} a_{km} = \frac{1}{\pi}\sum_{m=1}^{n}\left[h\cot\frac{x_m - \hat{x}_k}{2}\right] = 0, \tag{43}$$

from (43), we know that A_n is singular matrix, then we cannot use system (40) or (41) to solve the integral Equation (37).

In order to get a well-conditioned definite system, we introduce a regularizing factor γ_{0n} in (40), which leads to linear system

$$\begin{cases} \gamma_{0n} + \dfrac{1}{\pi}\displaystyle\sum_{m=1}^{n}\left[h\cot\frac{x_m - \hat{x}_k}{2}\right] f_m = g\left(\hat{x}_k\right), \\ \displaystyle\sum_{m=1}^{n} f_m = 0, \end{cases} \tag{44}$$

where γ_{0n} defined by

$$\gamma_{0n} = \frac{1}{2\pi}\sum_{k=1}^{n} g\left(\hat{x}_k\right) h. \tag{45}$$

Then the matrix form of system (44) can be presented as

$$A_{n+1} F_{n+1}^a = G_{n+1}^e, \tag{46}$$

where

$$A_{n+1} = \begin{pmatrix} 0 & e_n^{\mathrm{T}} \\ e_n & A_n \end{pmatrix}, F_{n+1}^a = \begin{pmatrix} \gamma_{0n} \\ F_n^a \end{pmatrix}, G_{n+1}^e = \begin{pmatrix} 0 \\ G_n^e \end{pmatrix}, \tag{47}$$

and $e_n = \Big(\underbrace{1, 1, \cdots, 1}_{n}\Big)^{\mathrm{T}}$.

Example 2: *Now we consider an example of solving Hilbert integral equation by collocation scheme. Let* $g\left(s\right) = \cos s - \sin s$, *the exact solution is* $f\left(x\right) = \cos x + \sin x$.

We examine the maximal nodal error, defined by

$$e_\infty = \min_{1\le i\le n}\left|f\left(x_i\right) - f_i\right|, \tag{48}$$

where $f_i\left(i = 1, 2, \cdots, n\right)$ denotes the approximation of $f\left(x\right)$ at $\hat{x}_i$. Numerical results presented in **Table 5** show that both the maximal nodal errors are as follow.

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Table 5. Errors for the solution of the Hilbert integral equation of first kind.

n	e_∞
32	2.2204e−16
64	1.9984e−15
128	3.9968e−15
256	8.8818e−15
512	1.4433e−14

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